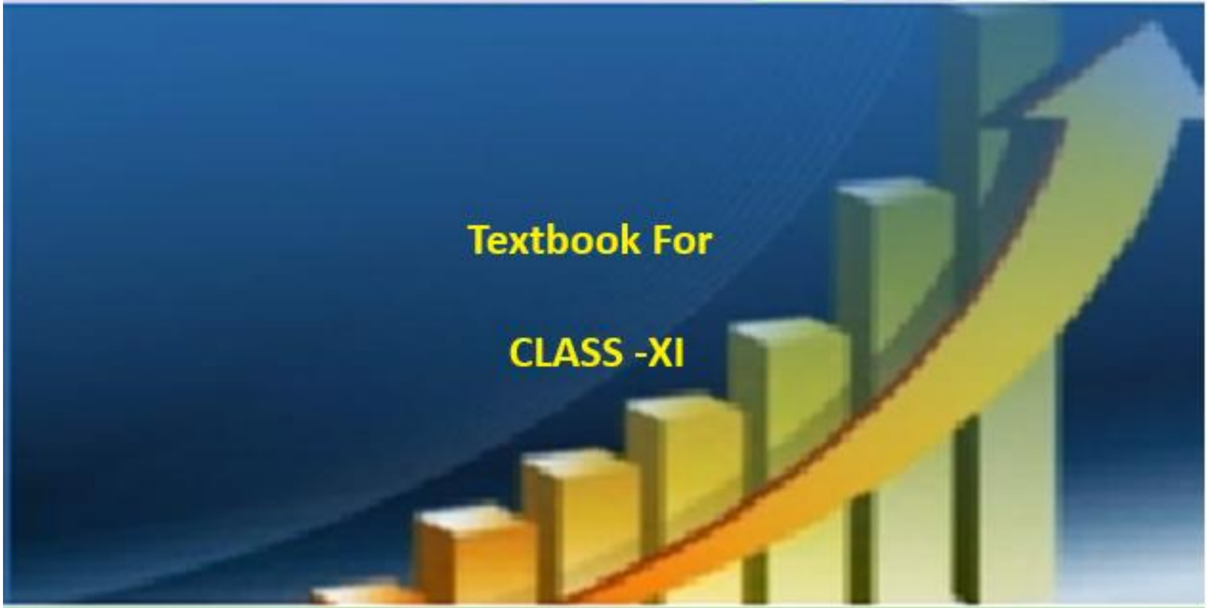
BUSINESS MATHEMATICS AND STATISTICS



Textbook For

CLASS -XI

ASSAM STATE SCHOOL EDUCATION BOARD, DIVISION- II
Bamunimaidam, Guwahati-21

Business Mathematics and Statistics

[For Higher Secondary First Year]

(As per revised syllabus of Assam State School Education
Board (ASSEB) First Year Course

(Effective from 2026-2027 Academic Session)

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BUSINESS MATHEMATICS AND STATISTICS

(Revised Syllabus for Higher Secondary First Year Course)

w.e.f. session 2026-2027

Theory : 80 Marks Time : Three Hours

Project : 20 Marks

Unit wise Distribution of Marks & Periods:

Units	Topics	Marks	Periods
Group-A	**Business Mathematics (40 Marks)**		
Unit-I	Financial Mathematics-I	10	20
Unit-II	Sets and Linear Functions	10	20
Unit-III	Indices and Logarithm	10	20
Unit-IV	Quadratic Equation and Linear Simultaneous Equation	10	20
Group-B	**Statistics (40 Marks)**		
Unit-I	Introduction to Statistics	10	20
Unit-II	Frequency Distribution; Diagrams and Graph	10	25
Unit-III	Measures of Central Tendency	20	25
	Total (Theory)	80	150
Group-C	**Project Work**	20	30
	Total (Theory + Project)	**100**	**180**

Group-A: BUSINESS MATHEMATICS (40 marks)

Unit-I : FINANCIAL MATHEMATICS-I **Marks-10**

Ratio and Proportion, Profit and Loss (Simple cases), Discount, Mixture

Unit-II : SETS AND LINEAR FUNCTIONS **Marks-10**

Definition of Sets, Operations on sets, Applications of sets, (Simple Problems), Definition of Function, Some functions related to Business and Economics.

Unit-III : INDICES AND LOGARITHM **Marks-10**

Laws of Indices, Logarithm and Antilogarithm, use of log tables.

Unit-IV : **QUADRATIC EQUATION AND LINEAR SIMULTANEOUS EQUATIONS** **Marks-10**

Linear simultaneous equations and its applications. Quadratic equation and its applications.

Group-B: STATISTICS (40 marks)

Unit-I : **INTRODUCTION TO STATISTICS** **Marks-10**

Meaning of Statistics, Importance and Scope of statistics in Business and Economics. Types of Data : Primary and Secondary. Methods of collection of Primary and Secondary data. Type of enquiry: Sample survey vs complete enumeration. Difference between Questionnaire and Schedule.

Unit-II : **FREQUENCY DISTRIBUTION: DIAGRAMS AND GRAPH** **Marks-10**

Basic concept of Frequency distribution with examples.

Different types of diagrams; Bar diagram (simple multiple and sub-divided and Pie diagram. Different types of graphs (Histogram, Frequency curve, Ogive and frequency polygon).

Unit-III : **MEASURES OF CENTRAL TENDENCY** **Marks-20**

Introduction and objectives of averages, Characteristics of a good average. Different measures of Central tendency: AM, GM, HM and their relations, Median, Partition, values and Mode.

Group - C : Project work (20 Marks)

Project Work -	**20 Marks**
■ Project Preparation	12 Marks
■ Project Viva Voce	08 Marks

Format for Project Work of the subject Business Mathematics and Statistics (H.S. First Year)

Cover Page :-

1. Title of the Project.
2. Information of the student
   (Name, Roll No., Registration No., Year)
3. Name of the Supervisor/Guide.
4. Name of the Institution.
5. Year

Second Page : Acknowledgement.

Third Page : Declaration by the students.

Fourth Page : Certificate from Supervisor/Guide.
Certificate from Head of the Institution/Department.

Fifth Page : Contents/Index.
Main text of the project
References/Bibliography

Introduction to population and sample. Types of Sampling methods. Preparation of a questionnaire. Statistical Investigation and its stage.

Students may prepare a project report on any one of the following pattern of topics:

a) Preparation of questionnaire of different types of case studies (students may prepare questionnaire for any one case study).

b) Assignment based project on planning stage of a statistical investigation.

c) Diagrammatic representation of case study data and its interpretation.

d) Graphical representation of data and its interpretation.

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Content

Preface

Mathematics and Statistics enhance logical thinking, analytical skills and problem-solving abilities. A strong foundation in Mathematics and Statistics open doors to various career paths. It also helps in students' future academic and professional endeavours.

As per Assam State School Education Board (ASSEB), the syllabus for H.S First year and H.S. Second year (Commerce) has undergone revision. The book has been renamed as "Business Mathematics and Statistics".

The present book makes an attempt to cover the theoretical, practical and applied aspects of Mathematics and Statistics. The chapters in this book are well arranged with objective-type, short-type and long-type questions to enable students to recapitulate their understanding in the subjects. Solved and unsolved illustrations problems are given in large number.

The Editor takes this opportunity to express gratitude to her co-writers Dr. Jnanjyoti Sarma, Mr. Bhaba Nanda Dutta, Mr. Ramen Ch. Deka, Dr. Kulojit Pathak and Mr. Amirul Islam for their contributions in preparing this textbook.

Suggestions regarding improvement of the book are most welcome and it would help to make this book more user-friendly in subsequent editions.

Editor

Group A: Business Mathematics

Unit-I : Financial Mathematics-I

Ratio and Proportion

Ratio

A ratio is a comparison of two quantities of the same kind. It is a relation that one quantity bears to another with respect to magnitude. In other words, ratio means how many times or what part one quantity is of the other.

Let a and b be two quantities of same kind expressed in same unit. Then the ratio of a and b is $\frac{a}{b}$ or $a \div b$ and is denoted by $a : b$. The two quantities that are being compared are called terms. The first is called antecedent and the second term is called consequent.

For example, the ratio 5 : 6 represents $\frac{5}{6}$ with antecedent 5 and consequent 6.

Types of Ratios

1. Inverse Ratio or Reciprocal Ratio

If the places of the antecedent and consequent of a ratio are interchanged, then the new ratio thus formed is called the inverse ratio of the given ratio. e.g. the ratio $b : a$ is the inverse ratio of $a : b$ and vice versa. i.e, the ratio $a : b$ and the ratio $b : a$ are inverse ratio of one another.

2. Compound Ratio

The ratio of the product of the antecedents to that of the consequents of two or more given ratios is called the compound ratio.

Thus if $a : b$ and $c : d$ are two given ratios, then $ac : bd$ is the compound ratio of the given ratios.

If 2 : 3, 3 : 4 and 4 : 5 be the given ratios, then their compound ratio will be

$$= (2\times3\times4):(3\times4\times5)$$

$$= \frac{2\times3\times4}{3\times4\times5} = \frac{2}{5} = 2:5$$

3. Duplicate Ratio

The ratio of the squares of two numbers is called the duplicate ratio of the ratio of the given numbers.

For example, $\frac{2^2}{3^2}$ or $\frac{4}{9}$ is the duplicate ratio of the ratio $\frac{2}{3}$.

4. Triplicate Ratio

The ratio of the cubes of two numbers is called the triplicate ratio of the ratio of the given numbers.

For example, $\frac{2^3}{3^3}$ or $\frac{8}{27}$ is the triplicate ratio of the ratio $\frac{2}{3}$.

5. Sub-duplicate Ratio

The ratio of the square roots of two numbers is called the sub-duplicate ratio of the ratio of the given numbers. For example, $\frac{2}{3}$ is the sub-duplicate ratio of the ratio $\frac{4}{9}$.

6. Sub-triplicate Ratio

The ratio of the cube roots of two numbers is called the sub-triplicate ratio of the ratio of the given numbers. For example, $\frac{2}{3}$ is the sub-triplicate ratio of the ratio $\frac{8}{27}$.

■ Proportion

When two ratios are equal, the quantities forming the ratio are said to be in proportion or proportional. If a, b, c, d are four quantities such that $a : b = c : d$, then we say that the quantities a, b, c, d are in proportion. The first and the last term i.e., a and d are called Extrems whereas b and c are called means. Again d is called the fourth proportional to a, b and c.

The terms of each ratio forming proportion should be of same kind but the terms of both the ratios may or may not be of same kind

e.g., 3 cm : 5 cm = Rs. 9 : Rs. 15.

Note: The proportion a : b = c : d is also expressed as $a : b :: c : d$.

■ Mean Proportional and Third Proportional

If three quantities are such that the ratio of first to the second is equal to the ratio of second to the third, then the second quantity is said to be the mean proportional between the first and the third and the third quantity is said to be the third proportional to the first and the second.

For example, consider the three numbers 9, 27 and 81.

$$9:27 = \frac{9}{27} = \frac{1}{3} = 1:3$$

$$27:81 = \frac{27}{81} = \frac{1}{3} = 1:3$$

Thus 9 : 27 = 27 : 81

$\therefore$ 27 is the mean proportional between 9 and 81 and 81 is the third proportional to 9 and 27.

■ Continued Proportion

When a number of quantities having same unit are such that the ratio of first and second quantities, second and third quantities, third and fourth quantities and so on, then these quantities are said to be in continued proportion.

For example, Rs. 3, Rs. 12, Rs. 48, Rs. 192, Rs. 768 are in continued proportion.

Since Rs. 3 : Rs. 12 = $\frac{3}{12}=\frac{1}{4}=1:4$

Rs. 12 : Rs. 48 = $\frac{12}{48}=\frac{1}{4}=1:4$

Rs. 48 : Rs. 192 = $\frac{48}{192}=\frac{1}{4}=1:4$

Rs. 192 : Rs. 768 = $\frac{192}{768}=\frac{1}{4}=1:4$

■ Laws of Proportional

1. Cross multiplication

   If $a : b = c : d$ then $ad = bc$
2. Invertendo

   If $a : b = c : d$ then $b : a = d : c$
3. Alternendo

   If $a : b = c : d$ then $a : c = b : d$
4. Componendo

   If $a : b = c : d$ then $a + b : b = c + d : d$
5. Dividendo

   If $a : b = c : d$ then $a - b : b = c - d : d$
6. Componendo and Dividendo

   If $a : b = c : d$ then $a + b : a - b = c + d : c - d$

Worked out Examples

Example 1 (a) : Find the fourth proportional to the numbers 60, 48, 30.

Solution: Let fourth proportional = x

$\therefore$ 60 : 48 = 30 : x

$$\Rightarrow \frac{60}{48}=\frac{30}{x}$$

$$\Rightarrow x=\frac{3\cancel{0}\times\cancel{48}^{8}}{\cancel{6}\cancel{0}}$$

$$\Rightarrow x=24$$

$\therefore$ The required fourth proportional = 24.

(b) Find the third proportional to 0.8 and 0.2.

Solution: Let third proportional = x

$\therefore\ 0.8 : 0.2 = 0.2 : x$

$$\Rightarrow \frac{0.8}{0.2} = \frac{0.2}{x}$$

$$\Rightarrow x = \frac{0.2 \times 0.2}{0.8}$$

$$= 0.05$$

$\therefore$ The required third proportional = 0.05.

(c) Find the mean proportional between 64 and 81.

Solution: Let x = Mean Proportional

$\therefore\ 64 : x = x : 81$

$$\Rightarrow \frac{64}{x} = \frac{x}{81}$$

$$\Rightarrow x^2 = 64 \times 81 = 8^2 \times 9^2$$

$$\Rightarrow x = \sqrt{8^2 \times 9^2}$$

$$= 8 \times 9$$

$$= 72$$

$\therefore$ The required mean proportion = 72.

Example 2 (a) : If $x : y = 8 : 9$, then find the value of $5x - 4y : 3x + 2y$.

Solution: $x : y = 8 : 9$

$$\Rightarrow \frac{x}{y} = \frac{8}{9}$$

$$\Rightarrow \frac{x}{8} = \frac{y}{9} = k \text{ (say)}$$

$$\therefore\ x = 8k,\ y = 9k$$

$$\therefore\ 5x - 4y : 3x + 2y$$

$$= \frac{5x - 4y}{3x + 2y}$$

$$=\frac{5(8k)-4\,(9k)}{3(8k)+2(9k)}$$

$$=\frac{40k-36k}{24k+18k}$$

$$=\frac{4k}{42k}$$

$$=\frac{2}{21}=2:21$$

$$\therefore 5x-4y:3x+2y=2:21$$

(b) If $a:5=b:7=c:8$ then prove that $\frac{a+b+c}{a}=4$

Solution: $a:5=b:7=c:8$

$$\Rightarrow \frac{a}{5}=\frac{b}{7}=\frac{c}{8}=k \text{ (say)}$$

$$\Rightarrow a=5k,\, b=7k,\, c=8k$$

$$\therefore \frac{a+b+c}{a}=\frac{5k+7k+8k}{5k}$$

$$=\frac{20k}{5k}$$

$$=4$$

(c) If A : B = 2 : 3 and B : C = 4 : 5 then find A : B : C

Solution: A : B = 2 : 3

$$\Rightarrow \frac{\mathrm{A}}{\mathrm{B}}=\frac{2}{3}=\frac{2\times 4}{3\times 4}=\frac{8}{12}$$

B : C = 4 : 5

$$\Rightarrow \frac{\mathrm{B}}{\mathrm{C}}=\frac{4}{5}=\frac{4\times 3}{5\times 3}=\frac{12}{15}$$

$\therefore$ A : B = 8 : 12 and B : C = 12 : 15

$\therefore$ A : B : C = 8 : 12 : 15.

(d) If $x+y:x-y=5:2$ find $x:y$

Solution: $x+y:x-y=5:2$

$$\Rightarrow \frac{x+y}{x-y} = \frac{5}{2}$$

$$\Rightarrow 5(x-y) = 2(x+y)$$

$$\Rightarrow 5x - 5y = 2x + 2y$$

$$\Rightarrow 5x - 2x = 5y + 2y$$

$$\Rightarrow 3x = 7y$$

$$\Rightarrow \frac{x}{y} = \frac{7}{3}$$

$$\Rightarrow x : y = 7 : 3$$

Example 3 : Two numbers are in the ratio of 4 : 5 and the sum of these numbers is 27. Find the two numbers.

Solution: Let the two numbers be x and y

$$\therefore\ x : y = 4 : 5$$

$$\Rightarrow \frac{x}{y} = \frac{4}{5} \Rightarrow 5x = 4y \quad \text{— (1)}$$

Also $x + y = 27$ — (2)

$$\Rightarrow x + \frac{5x}{4} = 27 \quad [\text{by (1) } 5x = 4y \Rightarrow y = \frac{5x}{4}\ ]$$

$$\Rightarrow \left(1 + \frac{5}{4}\right)x = 27$$

$$\Rightarrow \left(\frac{4+5}{4}\right)x = 27$$

$$\Rightarrow \frac{9}{4}x = 27$$

$$\Rightarrow x = \frac{\overset{3}{\cancel{27}} \times 4}{\cancel{9}}$$

$$\Rightarrow x = 3 \times 4 = 12$$

From (2) $y = 27 - x$

$= 27 - 12$

$= 15$

$\therefore$ The two numbers are 12 and 15.

Example 4 : Divide Rs. 670 among A, B and C in the ratio $\frac{2}{3}:\frac{1}{5}:\frac{1}{4}$.

Solution: Dividing ratio $=\frac{2}{3}:\frac{1}{5}:\frac{1}{4}$

$$=\frac{2}{3}\times 60:\frac{1}{5}\times 60:\frac{1}{4}:60 \qquad [\therefore \text{ L.C.M. of 3, 5, 4 = 60}]$$

$$=40:12:15$$

$\therefore$ A will receive = Rs. $\left(\frac{40}{40+12+15}\times 670\right)$

$$=\text{Rs.}\left(\frac{40}{67}\times 670\right)$$

= Rs. 400

B will receive = Rs. $\left(\frac{12}{67}\times 670\right)$

= Rs. 120

C will receive = Rs. $\left(\frac{15}{67}\times 670\right)$

= Rs. 150

Example 5 : The ratio between two numbers is 2 : 7. If each of them is increased by 14, the ratio between the new numbers obtained is 4 : 7. Find the original numbers.

Solution: Let the original numbers be x and y.

$$\therefore x:y=2:7$$

$$\Rightarrow \frac{x}{y}=\frac{2}{7}$$

$$\Rightarrow 7x=2y \quad \text{— (1)}$$

According to the question,

$$x+14:y+14=4:7$$

$$\Rightarrow \frac{x+14}{y+14}=\frac{4}{7}$$

$$\Rightarrow 7(x+14)=4(y+14)$$

$\Rightarrow 7x + 98 = 4y + 56$

$\Rightarrow 2y + 98 = 4y + 56$ [$\because$ $7x = 2y$ by (1)]

$\Rightarrow 4y - 2y = 98 - 56$

$$\Rightarrow 2y = 42 \Rightarrow y = \frac{42}{2} = 21$$

From (1) $7x = 2 \times 21$ ($\because y = 21$)

$$\Rightarrow x = \frac{2 \times \cancel{21}^{3}}{\cancel{7}} = 6$$

$\therefore$ The original numbers are 6 and 21.

Example 6 : Divide Rs. 798 among A, B, C and D such that A's share : B's share = 1 : 2; B's share : C's share = 3 : 4 and C's share : D's share = 5 : 6.

Solution: A : B = 1 : 2 $\Rightarrow$ A : B = 15 : 30

$$\text{B : C} = 3 : 4 \Rightarrow \frac{\text{B}}{\text{C}} = \frac{3}{4} = \frac{3 \times 10}{4 \times 10} = \frac{30}{40}$$

$\Rightarrow$ B : C = 30 : 40

$$\text{C : D} = 5 : 6 \Rightarrow \frac{\text{C}}{\text{D}} = \frac{5}{6} = \frac{5 \times 8}{6 \times 8} = \frac{40}{48}$$

$\therefore$ C : D = 40 : 48

$\therefore$ A : B : C : D = 15 : 30 : 40 : 48

Sum of the ratios = 15 + 30 + 40 + 48

= 133

$$\text{A's share} = \text{Rs.} \left(\frac{15}{\cancel{133}} \times \overset{6}{\cancel{798}}\right) = \text{Rs. } 90$$

$$\text{B's share} = \text{Rs.} \left(\frac{30}{\cancel{133}} \times \overset{6}{\cancel{798}}\right) = \text{Rs. } 180$$

$$\text{C's share} = \text{Rs.} \left(\frac{40}{\cancel{133}} \times \overset{6}{\cancel{798}}\right) = \text{Rs. } 240$$

$$\text{D's share} = \text{Rs.} \left(\frac{48}{\cancel{133}} \times \overset{6}{\cancel{798}}\right) = \text{Rs. } 288$$

Example 7 : Divide Rs. 490 among A, B and C such that

$$\frac{2}{3} \text{ of A's share} = \frac{3}{4} \text{ of B's share} = \frac{4}{5} \text{ of C's share.}$$

Solution: Let A's share = Rs. x

B's share = Rs. y

C's share = Rs. z

According to the question,

$$\frac{2}{3}x = \frac{3}{4}y = \frac{4}{5}z$$

$$\frac{2}{3}x = \frac{3}{4}y \Rightarrow \frac{x}{y} = \frac{9}{8} \quad \text{— (1)}$$

$$\frac{3}{4}y = \frac{4}{5}z \Rightarrow \frac{y}{z} = \frac{16}{15} \quad \text{— (2)}$$

From (1) $\frac{x}{y} = \frac{9}{8} = \frac{9\times 2}{8\times 2} = \frac{18}{16}$

i.e. $\frac{x}{y} = \frac{18}{16}$ — (3)

From (2) and (3) we get

$$x : y : z = 18 : 16 : 15$$

A's share = Rs. $\left(\frac{18}{18+16+15}\times 490\right)$

= Rs. $\left(\frac{18}{49}\times 490\right)$ = Rs. 180

B's share = Rs. $\left(\frac{16}{18+16+15}\times 490\right)$

= Rs. $\left(\frac{16}{49}\times 490\right)$ = Rs. 160

C's share = Rs. $\left(\frac{15}{18+16+15}\times 490\right)$ = Rs. 150

Example 8 : Divide Rs. 558 among Dipak, Rohit and Manish such that if Rs. 4, Rs. 6 and Rs. 8 are decreased respectively from their shares, then the ratio of their shares becomes 2 : 3 : 7.

Solution: Sum of the decreased amount of Dipak, Rohit and Manish = Rs. (4 + 6 + 8) = Rs. 18.

$\therefore$ Amount to be divided = Rs. (558 – 18)

= Rs. 540

$$\therefore \text{ Dipak will receive} = \text{Rs. } \left[\left(\frac{2}{2+3+7} \times 540\right) + 4\right]$$

$$= \text{Rs. } \left[\left(\frac{2}{\cancel{12}} \times \overset{45}{\cancel{540}}\right) + 4\right]$$

= Rs. (90 + 4)

= Rs. 94

$$\text{Rohit will receive} = \text{Rs. } \left[\left(\frac{3}{2+3+7} \times 540\right) + 6\right]$$

$$= \text{Rs. } \left[\left(\frac{3}{\cancel{12}} \times \overset{45}{\cancel{540}}\right) + 6\right]$$

= Rs. (135 + 6)

= Rs. 141

$$\text{Manish will receive} = \text{Rs. } \left[\left(\frac{7}{2+3+7} \times 540\right) + 8\right]$$

$$= \text{Rs. } \left[\left(\frac{7}{\cancel{12}} \times \overset{45}{\cancel{540}}\right) + 8\right]$$

= Rs. (315 + 8)

= Rs. 323

Example 9 : Men, women and children are employed in the ratio 1 : 2 : 3 to do a work and their daily wages are in the ratio 6 : 3 : 2. When 50 men are employed then total wages of all the workers amount to Rs. 9,000. Find the daily wages paid to 1 man, 1 woman and 1 child.

Solution: Men, women and children are employed in the ratio 1 : 2 : 3.

Let Men employed = x

women employed = $2x$

children employed = $3x$

Daily wages of men, women and children are in the ratio 6 : 3 : 2.

Let daily wage of 1 man = $6y$

daily wage of 1 woman = $3y$

daily wage of 1 child = $2y$

$\therefore$ Total wages of all the workers

$= (x)\ (6y) + (2x)\ (3y) + (3x)\ (2y)$

$= 6xy + 6xy + 6xy$

$= 18xy$

Given that total wages of all the workers is Rs. 9,000 when men employed = 50 i.e. $18xy = 9000$ when $x = 50$

$$\therefore\ 18 \times 50 \times y = 9000$$

$$\Rightarrow y = \frac{\overset{\cancel{50}}{\cancel{9000}}\,^{10}}{\cancel{18} \times \cancel{50}}$$

$$\therefore\ y = 10$$

$\therefore$ Daily wage of 1 man = Rs. (6×10) = Rs. 60

Daily wage of 1 woman = Rs. (3×10) = Rs. 30

Daily wage of 1 child = Rs. (10×2) = Rs. 20

Example 10 : A man divides his property among his son, wife and daughter in such a way that son's share to daughter's share is 8 : 7 and wife's share to son's share is also 8 : 7. If the wife receives Rs. 300 more than the daughter, find the share of each.

Solution: Let son's share = Rs. x

wife's share = Rs. y

daughter's share = Rs. z

According to the question,

$$x : z = 8 : 7$$

$$\Rightarrow \frac{x}{z} = \frac{8}{7} \quad \text{— (1)}$$

Also $y : x = 8 : 7$

$$\Rightarrow \frac{y}{x} = \frac{8}{7} \quad \text{— (2)}$$

Also $y = z + 300$ — (3)

From (1) and (2) we get

$$\frac{x}{z} \times \frac{y}{x} = \frac{8}{7} \times \frac{8}{7}$$

$$\Rightarrow \frac{y}{z} = \frac{64}{49}$$

$$\Rightarrow y = \left(\frac{64}{49}\right)z \quad \text{— (4)}$$

Now (3) $\Rightarrow y = z + 300$

$$\Rightarrow \left(\frac{64}{49}\right)z = z + 300 \text{ [by (4)]}$$

$$\Rightarrow \left(\frac{64}{49} - 1\right)z = 300$$

$$\Rightarrow \frac{15}{49}z = 300$$

$$\Rightarrow z = \frac{\overset{20}{\cancel{300}} \times 49}{\cancel{15}}$$

$$\Rightarrow z = 980$$

(3) $\Rightarrow y = z + 300$

$= 980 + 300$

$= 1280$ i.e. $y = 1280$

From (1), $x = \frac{8}{7}z = \frac{8}{\cancel{7}} \times \overset{140}{\cancel{980}} = 1120$

$\therefore$ Son's share = Rs. 1120

Wife's share = Rs. 1280

Daughter's share = Rs. 980

Example 11 : The railway fare in a certain year increases in the ratio 22 : 25 but the number of passengers decrease in the ratio 13 : 11. Find in what ratio will the total income from passenger's fare increase or decrease?

Solution: For the certain year

Let original railway fare = Rs. $22x$ per passenger

and the railway fare after the increase of fare = Rs. 25 per passenger

Let number of original passengers = $13y$

and total income from original passengers

= Rs. $(22x \times 13y)$

Also total present income from passengers = Rs. $(25x \times 11y)$

∴ Original income : Present income

$$= \text{Rs. } (22x \times 13y) : \text{Rs. } (25x \times 11y)$$

$$= \frac{\overset{2}{\cancel{22}}\cancel{x} \times 13\cancel{y}}{25\cancel{x} \times \cancel{11}\cancel{y}}$$

$$= \frac{26}{25} = 26 : 25$$

∴ Total income will decrease in the ratio 26 : 25.

Example 12 : 20 men can do a piece of work in 12 days. In how many days can 15 men finish the same work?

Solution: Let 15 men can do the work in x days.

	Men	Days	
Arrangement :	20	12	Less men will require more days
	15	x	

$$x = \overset{4}{\cancel{12}} \times \frac{\overset{4}{\cancel{20}}}{\cancel{15}_{\cancel{3}}} = 16 \text{ days}$$

[Since less men will require more days therefore write the greater ratio]

∴ 15 men will finish the work in 16 days.

Example 13 : 15 men working 10 days earn Rs. 500, how much will 12 men earn working 14 days?

Solution: Let 12 men working 14 days will earn Rs. x

Arrangement :	Men	Days	Income (Rs.)
	15	10	500
	12	14	x

$$x = \overset{10}{\cancel{500}} \times \frac{\overset{4}{\cancel{12}}}{\cancel{15}_{\cancel{3}}} \times \frac{14}{\cancel{10}}$$

= 560 **Ans:** Rs. 560

[**Note:** 12 men is less than 15 men. Less men will earn less therefore the lesser ratio $\frac{12}{15}$. Also 14 days is more than 10 days. More days indicates more income. Hence the greater ratio $\frac{14}{10}$]

Example 14 : 50 men working 8 hours a day earn Rs. 3264 in 21 days. How much will 35 men earn in 25 days working 14 hours a day?

Solution: Let required earn = Rs. x

Arrangement :	Men	Hour	Days	Earning
	50	8	21	3264
	35	14	25	x

$$x = \overset{\overset{136}{\cancel{408}}}{\cancel{3264}} \times \frac{35}{\underset{\cancel{2}}{\cancel{50}}} \times \frac{\overset{\cancel{2}}{\cancel{14}}}{8} \times \frac{\cancel{25}}{\cancel{21}_{\cancel{3}}}$$

$$= 136 \times 35 = 4760$$

Ans: Rs. 4760

[**Note:** Less men will earn less, so lesser ratio $\frac{35}{50}$. More hour means more earning, so greater ratio $\frac{14}{8}$. More days means more earn, so greater ratio $\frac{25}{21}$]

Example 15 : A contractor undertakes to complete a work by 100 days. He employed 160 men who worked 6 hours a day finished $\frac{2}{3}$ of the work in 80 days. How many more men should he employ to finish the work within time if the men work for 8 hours a day?

Solution: Let additional men employed = x

Remaining days = (100 – 80) days = 20 days

Remaining work $= 1 - \frac{2}{3} = \frac{1}{3}$

Arrangement :	Days	Hour	Work	Men
	80	6	$\frac{2}{3}$	160
	20	8	$\frac{1}{3}$	$160 + x$

$$\therefore\ 160 + x = 160 \times \frac{8\cancel{0}}{2\cancel{0}} \times \frac{{}^{3}\cancel{6}}{\cancel{8}} \times \frac{1/3}{2/3}$$

$$= \overset{80}{\cancel{160}} \times 3 \times \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}}$$

$$= 240$$

Less Days ⇒ More men
More Hour ⇒ Less men
Less work ⇒ Less men

$\Rightarrow x = 240 - 160 = 80$

$\therefore$ 80 more men should be employed.

Example 16 : If 5 men, 20 women and 40 boys can do a piece of work in 120 days, working 6 hours a day, in how many days will 10 men, 10 women and 20 boys, do the same work working 9 hours a day when amount of work done by man,woman and boy are in the ratio 4 : 2 :1?

Solution: We have 1 man = 4 boys

1 woman = 2 boys

$\therefore$ 5 men + 20 women + 40 boys

$= (5 \times 4 + 20 \times 2 + 40)$ boys

= (20 + 40 + 40) boys = 100 boys

10 men + 10 women + 20 boys

$= (10 \times 4 + 10 \times 2 + 20)$ boys

= (40 + 20 + 20) boys = 80 boys

Arrangement	Boys	Hours	Days	
	100	6	120	Less boys $\Rightarrow$ More days
	80	9	x	More hour $\Rightarrow$ Less days

$$x = \overset{\overset{5}{\cancel{40}}}{\cancel{120}} \times \frac{10\cancel{0}}{\cancel{8}\cancel{0}} \times \frac{\overset{2}{\cancel{6}}}{\underset{\cancel{3}}{\cancel{9}}} = 100$$

Ans: 100 days

Example 17 : 8 men or 12 boys can do a piece of work in 20 days; in how many days can 6 men and 6 boys together do the same piece ofo work?

Solution: 8 men $\equiv$ 12 boys

$$\Rightarrow 2 \text{ men} \equiv \frac{12}{4} \text{ boys}$$

$$\Rightarrow 6 \text{ men} \equiv \left(3 \times \frac{\overset{3}{\cancel{12}}}{\cancel{4}}\right) \text{ boys}$$

$\therefore$ 6 men + 6 boys = (9 + 6) boys

= 15 boys

Let in x days 6 men and 6 boys together can do the piece of work.

Arrangement	Boys	Days	
	12	20	
	15	x	More boys $\Rightarrow$ Less days

$$x = \overset{4}{\cancel{20}} \times \frac{\overset{4}{\cancel{12}}}{\underset{\cancel{3}}{\cancel{15}}}$$ **Ans:** 16 days.

Example 18 : A can do a piece of work in 8 days, B can do the same work in 12 days. In how many days, can A and B together, do the same work?

Solution: A can do the work in 8 days

$\therefore$ In one day A can do $\frac{1}{8}$ parts of the work

B can do the work in 12 days

$\therefore$ In one day B can do $\frac{1}{12}$ parts of the work

$\therefore$ In one day A and B together can do $\left(\frac{1}{8}+\frac{1}{12}\right)$ parts of the work

i.e. $\left(\frac{3+2}{24}\right)$ parts of the work

i.e. $\frac{5}{24}$ parts of the work

$\therefore$ A and B together can finish the work in $\frac{24}{5}$ days i.e. in $4\frac{4}{5}$ days.

Example 19 : A can do a piece of work in 12 days, while B can do in 24 days. They begin to work together, but after 4 days B leaves, in how many days more, will A alone do the unfinished work?

Solution: A can do the work in 12 days.

$\therefore$ In one day A can do $\frac{1}{12}$ parts of the work

B can do the work in 24 days

$\therefore$ In one day B can do $\frac{1}{24}$ parts of the work

In one day A and B together can do $\left(\frac{1}{12}+\frac{1}{24}\right)$ parts of the work

i.e. $\left(\frac{2+1}{24}\right)$ parts of the work

i.e. $\frac{1}{8}$ parts of the work

In 4 days A and B together can do $\frac{4}{8}=\frac{1}{2}$ parts of the work

After 4 days B leaves

$\therefore$ After 4 days remaining work $= 1 - \frac{1}{2} = \frac{1}{2}$ parts of the work

To finish the whole work A takes 12 days

$\therefore$ To finish ½ part of the work A takes $= \left(\frac{1}{2} \times 12\right)$ days

$= 6$ days

$\therefore$ A can do the unfinished work in 6 days.

Example 20 : A and B together can do a piece of work in 15 days. They work together for 8 days when A leaves and B finishes the work in 15 days more. In how many days can A alone finish the piece of work?

Solution: A and B together can do the work in 15 days.

In one day A and B together can do $\frac{1}{15}$ parts of the work.

In 8 days A and B together have done $\frac{8}{15}$ parts of the work.

After 8 days remaining work $= 1 - \frac{8}{15} = \frac{7}{15}$ parts.

Now B finishes $\frac{7}{15}$ parts of the work in 15 days

$\therefore$ In one day B finishes $\frac{7}{15 \times 15} = \frac{7}{225}$ part of the work

$\therefore$ In one day A alone can finish $\left(\frac{1}{15} - \frac{7}{225}\right)$ parts of the work

i.e. $\frac{8}{225}$ parts of the work

$\therefore$ A alone can finish the work in $\frac{225}{8}$ days $= 28\frac{1}{8}$ days.

Ans : $= 28\frac{1}{8}$ days.

Example 21 : Arun and Rakesh can do a piece of work in 12 days, Rakesh and Neeraj together can do it in 15 days. If workman Arun is twice as good as Neeraj, find at what time Rakesh alone can do it.

Solution: As a workman Arun $\equiv 2$ (Neeraj)

Arun and Rakesh can do the work in 12 days

$\therefore$ In one day Arun and Rakesh can do $\frac{1}{12}$ parts of the work

$\therefore$ In one day, {2 (Neeraj) + Rakesh} can do $\frac{1}{12}$ parts of the work — (i)

Again Rakesh and Neeraj together can do the work in 15 days

$\therefore$ In one day (Neeraj + Rakesh) can do $\frac{1}{15}$ parts of the work — (ii)

Subtracting (ii) from (i) we get

In one day, Neeraj can do $\left(\frac{1}{12}-\frac{1}{15}\right)$ parts of the work

i.e. $\left(\frac{5-4}{60}\right)$ parts of the work

i.e. $\frac{1}{60}$ parts of the work.

Since as a workman Arun $\equiv 2$ (Neeraj)

$\therefore$ In one day Arun can do $\frac{2}{60}=\frac{1}{30}$ part of the work

But in one day, Arun and Rakesh together can do $\frac{1}{12}$ part of the work

$\therefore$ In one day Rakesh can do $\left(\frac{1}{12}-\frac{1}{30}\right)=\frac{5-2}{60}=\frac{3}{60}=\frac{1}{20}$ part of the work

$\therefore$ Rakesh alone can finish the work in 20 days.

Example 22 : A tap can fill a cistern in 8 hours and another can empty it in 16 hours. If both the taps are opened simultaneously, find the time (in hours) to fill the cistern.

Solution: The first tap can fill the cistern in 8 hours

$\therefore$ In 1 hour the first tap can fill $\frac{1}{8}$ part of the cistern.

The second tap can empty the cistern in 16 hours

$\therefore$ In 1 hour the second tap can empty $\frac{1}{16}$ part of the cistern

If both the taps are opened simultaneously, then in 1 hour, $\frac{1}{8}-\frac{1}{16}=\frac{2-1}{16}=\frac{1}{16}$ part of the cistern will be filled

$\therefore$ If both the cistern are opened simultaneously, then the cistern will be filled in 16 hours.

Example 23 :Two taps A and B can fill a cistern in 20 and 30 minutes respectively. Both the pipes being opened, find when the tap A must be turned off so that the cistern may be filled in 10 minutes more.

Solution: Tap A can fill the cistern in 20 minutes

$\therefore$ In 1 minute tap A can fill $\frac{1}{20}$ part of the cistern.

Tap B can fill the cistern in 30 minutes

$\therefore$ In 1 minute tap B can fill $\frac{1}{30}$ part of the cistern

When both the taps A and B are opened, then in 1 minute (tap A + tap B) can fill $\left(\frac{1}{20}+\frac{1}{30}\right)$ part of the cistern

i.e. $\left(\frac{3+2}{60}\right)$ part of the cistern

i.e. $\frac{5}{60}$ part of the cistern

i.e. $\frac{1}{12}$ part of the cistern.

Thus both the taps (tap A and tap B) together can fill the cistern in 12 minutes.

Now, when the tap A is turned off, the remaining part of the cistern is filled by tap B in 10 minutes more.

In one minute tap B can fill $\frac{1}{30}$ part of the cistern.

$\therefore$ In 10 minutes tap B can fill $\frac{10}{30}$ part $=\frac{1}{3}$ part of the cistern.

This means that the first $\left(1-\frac{1}{3}\right)$ part $=\frac{2}{3}$ part of the cistern were filled by the taps A and B together.

Tap A and tap B together can fill the cistern in 12 minutes.

$\therefore$ Tap A and tap B together can fill $\frac{2}{3}$ part of the cistern in $\left(\frac{2}{3}\times 12\right)$ minutes = 8 minutes.

$\therefore$ Tap A must be turned off after 8 minutes.

Example 24 : A tap can fill a tank in 6 hours. After half the tank is filled, three more similar taps are opened. What is the total time taken to fill the tank completely?

Solution: Half of the tank is filled in $\frac{1}{2}\times 6 = 3$ hours

Now we have four taps and each tap can fill the tank in 6 hours

When all the four taps are opened, then these four taps can fill ½ of the tank in $\frac{1}{2}\times\frac{6}{4}$ hours

$= \frac{3}{4}$ hours

$= \left(\frac{3}{4}\times 60\right)$ minutes

$= 45$ minutes

$\therefore$ Total time taken = 3 hours 45 minutes.

Example 25 : Two taps A and B can fill a cistern in 30 and 36 minutes respectively. Tap C can empty it at the rate of 50 litres per minute. If all the three taps are opened simultaneously, the cistern gets filled in 20 minutes. Find the capacity of the cistern.

Solution: Let capacity of the cistern = x litres

Tap A can fill the cistern in 30 minutes

$\therefore$ In one minute, tap A can fill $\frac{1}{30}$ part of the cistern.

$\therefore$ In 20 minutes tap A can fill $\frac{20}{30}$ part of the cistern.

$= \frac{2}{3}$ part of the cistern.

$\therefore$ In 20 minutes tap A fills the cistern $= \frac{2x}{3}$ litres

Tap B can fill the cistern in 36 minutes

$\therefore$ In one minute, tap B can fill $\frac{1}{36}$ part of the cistern

In 20 minutes tap B can fill $\frac{\overset{5}{\cancel{20}}}{\cancel{36}_9} = \frac{5}{9}$ part of the cistern.

In 20 minutes tap B fills the cistern $= \frac{5x}{9}$ litres

In 1 minute tap C can empty the cistern = 50 litres

In 20 minutes tap C can empty the cistern $= (50 \times 20)$ litres

$= 1000$ litres

When all the three taps are opened simultaneously, then we must have

$$\frac{2x}{3} + \frac{5x}{9} - 1000 = x$$

$$\Rightarrow \frac{2x}{3} + \frac{5x}{9} - x = 1000$$

$$\Rightarrow \frac{6x + 5x - 9x}{9} = 1000$$

$$\Rightarrow \frac{2x}{9} = 1000$$

$$\Rightarrow x = \frac{9}{\cancel{2}} \times \overset{500}{\cancel{1000}} = 4500$$

$\therefore$ Capacity of the cistern = 4500 litres.

Exercise

1 (i) Find the fourth proportional to 0.2, 0.12 and 0.3

(ii) Find the mean proportional of 0.25 and 0.04.

(iii) Find the third proportional to Rs. 3 and Rs. 1.20.

2. (i) If $(a+b):(a-b)=11:7$ find $a : b$

(ii) If $a : b = 3 : 4$ and $c : b = 4 : 5$ find $a : c$

(iii) If $x : 5 : 12 = 3 : y : 6$ find $x : y$

3. A and B earn in the ratio 2 : 1, spend in the ratio 5 : 3 and save in the ratio 4 : 1. If their combined monthly savings is Rs. 5,000, find the monthly income of each of them.

4. Divide Rs. 1674 among Moon, Dhiraj and Bhargab such that if Rs. 42, Rs. 23 and Rs. 9 are respectively deducted from their shares, then the ratio of their shares becomes 3 : 5 : 12.

5. 15 men or 12 women can complete a piece of work in 44 days. In how many days 4 men and 16 women can complete the same work?

6. 2 men and 5 boys can do ½ of a piece of work in 5 days. Again 3 men and 4 boys can do $\frac{1}{3}$ of the work in 3 days. How many days will 9 men take to complete the work?

7. A contractor undertakes to complete a work in 90 days. He first employed 152 men who worked 6 hours a day and finished $\frac{3}{4}$ of the work in 75 days. How many more men should he employ to finish the work in stipulated time if they work for 8 hours a day?

8. Prabhat finished $\frac{3}{5}th$ of a work in 9 days and the remaining work be finished in 4 days with the assistance of Rajib. Find in how many days Rajib alone can do the work.

9. A and B together can do a piece of work in 12 days, B and C in 15 days, and C and A in 20 days. How many days will A alone take to do the work?

10. A and B working together can complete a piece of work in 12 days and B and C working together can complete the same work in 16 days. A worked at it for 5 days and B worked at it for 7 days. C finished the remaining work in 13 days. How many days would C alone take to complete the work?

11. A water tank is $\frac{2}{5}$th full. Pipe A can fill the tank in 10 minutes and the Pipe B can empty it in 6 minutes. If both the pipes are opened, how long will it take to empty or fill the tank completely?

12. Two tapes A and B can separately fill a cistern in 10 minutes and 15 minutes respectively and when the waste pipe C is opened, then all three pipes A,B and C together can fill it in 18 minutes. Find the time to empty the cistern by the waste pipe C.

Answers

1 (i) 0.18 (ii) 0.1 (iii) Rs. 0.48 = 48 p

2 (i) 9 : 2 (ii) 15 : 16 (iii) 12 : 5

3. Rs. 14,000, Rs. 7,000

4. Moon = Rs. 282, Dhiraj = Rs. 423, Bhargab = Rs. 969

5. 24 days 6. 5 days 7. 38 men 8. 30 days 9. 30 days

10. 24 days 11. 6 minutes to empty 12. 9 minutes.

Profit and Loss

Introduction :

When a person deals in the purchase and sale of any item, he either gains or loses some amount generally. The commonly used terms in dealing with questions involving sale and purchase are cost price, selling price, profit, loss etc.

Cost Price : The price at which an article is purchased, is called the cost price of the article. Cost price is abbreviated as C.P.

Selling Price : The price at which an article is sold, is called the selling price of the article. Selling price is abbreviated as S.P.

Profit or Gain : If the selling price of an article is more than the cost price, then there is a gain or profit.

Profit or Gain = S.P. – C.P.

Loss : If the cost price of an article is more than the selling price, then there is a loss

Loss = C.P. – S.P.

Note that profit and loss are always calculated on cost price.

Profit percent : Profit percent is written as profit % and we use the following formula to find the profit percent

$$\boxed{\text{Profit \%} = \frac{\text{Profit}}{\text{C.P}} \times 100}$$

Loss percent : We write Loss Percent as Loss % and $\boxed{\text{Loss \%} = \frac{\text{Loss}}{\text{C.P}} \times 100}$

Note : However, if it is mentioned that we need to calculate profit or loss percent on the selling price, in that case

$$\boxed{\begin{gathered}\text{Profit \%} = \frac{\text{Profit}}{\text{S.P}} \times 100 \\ \text{Loss \%} = \frac{\text{Loss}}{\text{S.P}} \times 100\end{gathered}}$$

In such cases, x% profit or loss would mean that there is a profit or loss of Rs. x when S.P. = Rs. 100.

Worked out Examples

Example 1 : Mohan buys a watch for Rs. 350 and sells it for Rs. 392. Find his percentage of profit.

Solution: Here C.P = Rs. 350

S.P = Rs. 392

Profit = S.P – C.P

= Rs. (392 – 350) = Rs. 42.

$$\therefore \text{ Percentage of profit} = \frac{\text{Profit}}{\text{C.P}} \times 100\%$$

$$= \left(\frac{\overset{6}{\cancel{42}}}{\underset{7}{\cancel{350}}} \times \overset{2}{\cancel{100}}\right)$$

$$= 12\%$$

Example 2 : Show that

25% profit on C.P. = 20% profit on S.P.

Solution: 25% profit on C.P. means that if C.P. = Rs. 100 then S.P. = Rs. 125.

Let x be the		
C.P when	S.P.	C.P
S.P = Rs. 100	125	100
	100	x

$$\therefore \; x = \overset{20}{\cancel{100}} \times \frac{\cancel{100}^{4}}{\cancel{125}_{5}} = 80$$

Thus when S.P. = Rs. 100, C.P = Rs. 80

∴ Profit = S.P – C.P = Rs. (100 – 80) = Rs. 20

Thus profit = Rs. 20 when S.P = Rs. 100.

∴ Profit percent on S.P. = 20%

Thus 25% profit on C.P = 20% profit on S.P.

Example 3 : Suresh buys a camera for Rs. 1800 and sells it at 10% loss. Find its selling price.

Solution: Here C.P. = Rs. 1800

Loss = 10% of C.P.

$$= \left(\frac{10}{1\cancel{00}} \times 18\cancel{00}\right)$$

= Rs. 180

∴ S.P. = C.P. – Loss

= Rs. (1800 – 180)

= Rs. 1620.

Example 4 : A shopkeeper losses 7% by selling a cricket ball for Rs. 31. For how much should he sell the ball so as to gain 5% ?

Solution: 7% loss means if C.P. = Rs. 100 then S.P. = Rs. 93.

Let Rs. x be the C.P. of the ball

S.P.	C.P.
93	100
31	x

$$x = \text{Rs.}\left(100\times\frac{\cancel{31}}{{}_{3}\cancel{93}}\right)$$

$$= \text{Rs.}\frac{100}{3}$$

$\therefore$ C.P. of the cricket ball = Rs. $\frac{100}{3}$

5% gain means if C.P. = Rs. 100 then S.P. = Rs. 105

Let Rs. y be the S.P. of the cricket ball

When its C.P. = Rs. $\frac{100}{3}$

C.P.	S.P.
100	105
$\frac{100}{3}$	y

$$\therefore y = \text{Rs.}\left(105\times\frac{\frac{100}{3}}{100}\right)$$

$$= \text{Rs.}\left(\overset{35}{\cancel{105}}\times\frac{\cancel{100}}{3}\times\frac{1}{\cancel{100}}\right)$$

$$= \text{Rs. } 35$$

$\therefore$ To gain 5% he should sell the ball for Rs. 35.

Example 5 : When the selling price and gain% are given then express cost price in terms of gain% and selling price.

Or

Establish the formula

$$\text{C.P.} = \left(\frac{100}{100+\text{gain\%}}\right)\times \text{S.P.}$$

Solution: We have

$$\text{Gain\%} = \frac{\text{gain}}{\text{C.P}}\times 100$$

$$=\left(\frac{\text{S.P.}-\text{C.P.}}{\text{C.P}}\right)\times 100$$

$$=\left(\frac{\text{S.P.}}{\text{C.P.}}-1\right)\times 100$$

$$\Rightarrow \frac{\text{gain\%}}{100}=\frac{\text{S.P.}}{\text{C.P.}}-1$$

$$\Rightarrow \frac{\text{S.P.}}{\text{C.P.}}=1+\frac{\text{gain\%}}{100}$$

$$\Rightarrow \frac{\text{S.P.}}{\text{C.P.}}=\left(\frac{100+\text{gain\%}}{100}\right)$$

$$\Rightarrow \text{C.P}=\left(\frac{100}{100+\text{gain\%}}\right)\times \text{S.P.}$$

Note : From the above formula we have

$$\text{S.P.}=\left(\frac{100+\text{gain\%}}{100}\right)\times \text{C.P.}$$

Example 6 : If the cost price of 72 oranges is the same as the selling price of 64 oranges, find the gain percent.

Solution: C.P. of 72 oranges = S.P. of 64 oranges

Clearly, gain = S.P. of (72 – 64) = 8 oranges

Let S.P. of 8 oranges = Rs. x

$\therefore$ S.P. of $8\times 8=64$ oranges = Rs. $8x$

$\therefore$ C.P. of 72 oranges = Rs. $8x$

$\therefore$ Gain percent = $\frac{\text{gain}}{\text{C.P.}}\times 100\%$

$$=\left(\frac{\text{Rs.}x}{\text{Rs.}8x}\times 100\right)\%$$

$$=\left(\frac{1}{8}\times 100\right)\%$$

$$= 12.5\%$$

Example 7 : A man buys 4 lemons for Rs. 3 and sells 5 for Rs. 4. How much percent gain or loss does he make on the selling price?

Solution: Let total number of lemons bought = x

$$\text{C.P. of 4 lemons} = \text{Rs. } 3$$

$$\text{C.P. of } x \text{ lemons} = \text{Rs. } \frac{3x}{4}$$

$$\text{S.P. of 5 lemons} = \text{Rs. } 4$$

$$\text{S.P. of } x \text{ lemons} = \text{Rs. } \frac{4x}{5}$$

$$\frac{4x}{5} - \frac{3x}{4} = \frac{16x - 15x}{20} = \frac{x}{20} > 0$$

$$\therefore \text{ gain} = \frac{4x}{5} - \frac{3x}{4} = \frac{x}{20}$$

$$\therefore \text{ gain percent on S.P.} = \frac{\text{gain}}{\text{S.P.}} \times 100\%$$

$$= \left(\frac{\frac{x}{20}}{\frac{4x}{5}} \times 100 \right) \%$$

$$= \frac{\cancel{x}}{\underset{4}{\cancel{20}}} \times \frac{\cancel{5}}{4\cancel{x}} \times \overset{25}{\cancel{100}}\%$$

$$= \frac{25}{4}\% = 6\frac{1}{4}\%$$

Example 8 : A shopkeeper sold two bicycles for Rs. 1500 each. On one he gains 25% and on the other he losses 20%. What is his gain or loss percent in the whole transaction?

Solution: 25% profit means if C.P. = Rs. 100 then S.P. = Rs. 125.

Let Rs. x be the C.P. when S.P. = Rs. 1500

S.P.	C.P.
125	100
1500	x

$$x = \text{Rs.} \left(\overset{4}{100} \times \frac{\overset{300}{1500}}{\underset{5}{125}} \right)$$

$$= \text{Rs. } 1200$$

$\therefore$ C.P. of the first bicycle = Rs. 1200

20% loss means if C.P. = Rs. 100 then S.P. = Rs. 80

Let Rs. y be the C.P. when S.P. = Rs. 1500

S.P.	C.P.
80	100
1500	y

$$y = \text{Rs.}\left(\overset{25}{\cancel{100}}\times\frac{\overset{75}{\cancel{1500}}}{\underset{2}{\cancel{80}}}\right)$$

$$= \text{Rs. } 1875$$

$\therefore$ Total C.P. = Rs. (1200 + 1875) = Rs. 3075.

Total S.P. = Rs. (2×1500) = Rs. 3000

Clearly C.P. > S.P.

$\therefore$ There is a loss

Loss = C.P. – S.P.

= Rs. (3075 – 3000)

= Rs. 75

$$\therefore \text{ Loss percent } = \left(\frac{\text{Loss}}{\text{C.P.}}\times100\right)\%$$

$$= \left(\frac{\cancel{75}^{\cancel{1}}}{\underset{\cancel{123}\;41}{\cancel{3075}}}\times100\right)\%$$

$$= \frac{100}{41}\% = 2\frac{18}{41}\%\,.$$

Ans: $2\frac{18}{41}\%$ loss.

Example 9 : A person purchases 90 clocks and sells 40 clocks at a gain of 10% and 50 clocks at a gain of 20%. Had he sold all the clocks at a uniform profit of 15%, he would have got Rs. 40 less. Find the C.P. of each clock.

Solution: Let C.P. of each clock = Rs. x

S.P. of 40 clocks at a gain of 10%

= Rs. $40x$ + 10% of $40x$ [S.P. = C.P. + gain]

$$= \text{Rs. }\left(40x+\frac{1\cancel{0}}{1\cancel{0}\cancel{0}}\times4\cancel{0}x\right)$$

= Rs. $44x$

S.P. of 50 clocks at a gain of 20%

= Rs. $50x$ + 20% of $50x$

$$= \text{Rs.}\left(50x+\frac{2\cancel{0}}{1\cancel{0}\cancel{0}}\times 5\cancel{0}x\right)$$

= Rs. $(50x + 10x)$ = Rs. $60x$

∴ Total selling price of 90 clocks

= Rs. $44x$ + Rs. $60x$ = Rs. $104x$

S.P. of all the 90 clocks at a uniform profit of 15%

= Rs. $90x$ + 15% of $90x$

$$= \text{Rs.}\left(90x+\frac{\cancel{15}^{\,3}}{\underset{2}{\cancel{10}\cancel{0}}}\times 9\cancel{0}x\right)$$

$$= \text{Rs.}\left(90x+\frac{27x}{2}\right)= \text{Rs.}\left(\frac{180x+27x}{2}\right)$$

$$= \text{Rs. }\frac{207x}{2}$$

According to the question,

$$\frac{207x}{2}+40=104x$$

$$\Rightarrow\left(104-\frac{207}{2}\right)x=40$$

$$\Rightarrow\left(\frac{208-207}{2}\right)x=40$$

$$\Rightarrow x=2\times 40=80$$

∴ C.P. of each clock = Rs. 80.

Example 10 : If 12 eggs are bought for Rs. 10 and sold 10 for Rs. 12. What is the gain or loss% ?

Solution: Let number of eggs bought = x

C.P. of 12 eggs = Rs. 10

∴ C.P. of x eggs = Rs. $\frac{10x}{12}$ = Rs. $\frac{5x}{6}$

S.P. of 10 eggs = Rs. 12

S.P. of x eggs = Rs. $\frac{12x}{10}$ = Rs. $\frac{5x}{6}$

Now $\frac{6x}{5}-\frac{5x}{6}=\frac{36x-25x}{30}=\frac{11x}{30}>0$

$\therefore$ There is a gain.

gain = S.P. – C.P. = $\frac{6x}{5}-\frac{5x}{6}=\frac{11x}{30}$

gain% = $\frac{\text{gain}}{\text{C.P}}\times 100\%$

$$=\frac{\frac{11x}{30}}{\frac{5x}{6}}\times 100\%$$

$$=\frac{11x}{\cancel{30}}\times\frac{\cancel{6}^{2}}{\cancel{5x}}\times\overset{2}{\cancel{100}}\%$$

$= 44\%$ **Ans:** 44% gain

Example 11: The ratio of the cost price and the selling price is 4:5. What is the profit percent?

Solution: C.P. : S.P. = 4:5

i.e. $\frac{\text{C.P.}}{\text{S.P.}}=\frac{4}{5}$ — (1)

Profit percent = $\frac{\text{Profit}}{\text{C.P.}}\times 100\%$

$$=\left(\frac{\text{S.P}-\text{C.P}}{\text{C.P}}\right)\times 100\%$$

$$=\left(\frac{\text{S.P}}{\text{C.P}}-1\right)\times 100\%$$

$$=\left(\frac{5}{4}-1\right)\times 100\%$$

$=\frac{1}{4}\times 100\% = 25\%$ **Ans:** 25%

Example 12: A man sells his typewriter at 5% loss. If he sells it for Rs. 300 more, he will gain 5%. Find the cost price of the typewriter.

Solution: Method - 1

Let C.P = Rs. x

At 5% loss $\therefore$ S.P = C.P – 5% of C.P

$$= x - \frac{5}{100} \times x = \left(1 - \frac{1}{20}\right)x$$

$$= \frac{19}{20}x$$

At 5% gain S.P = C.P + gain

$$= x + 5\% \text{ of } x$$

$$= x + \frac{5x}{100} = \left(1 + \frac{1}{20}\right)x$$

$$= \frac{21}{20}x$$

According to the question,

$$\frac{19x}{20} + 300 = \frac{21}{20}x$$

$$\Rightarrow \frac{21}{20}x - \frac{19}{20}x = 300$$

$$\Rightarrow \frac{21x - 19x}{20} = 300$$

$$\Rightarrow \frac{2x}{20} = 300 \Rightarrow x = 3000$$

$\therefore$ Required C.P = Rs. 3000.

Method-2

Let C.P. = Rs. 100

5% loss mean S.P = Rs. 95

5% gain means S.P = Rs. 105

Difference of the two S.P. = Rs. 105 – Rs. 95

= Rs. 10

Difference	C.P.
10	100
300	?

Required C.P = Rs. $\left(100\times\frac{300}{10}\right)$

= Rs. 3000.

Example 13: A manufacturer sells a mobile phone to a wholesale dealer at a profit of 10%. The wholesaler sells the same to the retailer at a profit of 20%. The retailer, in turn sells the mobile phone to the customer for Rs. 6600 and thereby gains 25%. Find the cost price of the mobile.

Solution: Method - 1

Let the cost price of the mobile = Rs. x

Manufacturer gets 10% profit

S.P. for the manufacturer = Rs. $(x + 10\% \text{ of } x)$

$$= \text{Rs. } \left(x+\frac{10}{100}\times x\right)$$

$$= \text{Rs. } \frac{11x}{10}$$

Wholesaler gets 20% profit.

$$\therefore \text{ S.P. for the wholesaler } = \text{Rs. } \left(\frac{11x}{10}+20\% \text{ of} \frac{11x}{10}\right)$$

$$= \text{Rs. } \left(\frac{11x}{10}+\frac{\cancel{20}}{\underset{50}{\cancel{100}}}\times\frac{11x}{\cancel{10}}\right)$$

$$= \text{Rs. } \left(\frac{55x+11x}{50}\right)$$

$$= \text{Rs. } \frac{66x}{50}$$

S.P. for the wholesaler = C.P for the retailer

$$= \text{Rs. } \frac{66x}{50}$$

$$\text{S.P. for the retailer } = \text{Rs. } \left(\frac{66x}{50}+25\% \text{ of } \frac{66x}{50}\right)$$

$$= \text{Rs. } \left(\frac{66x}{50}+\frac{\cancel{25}}{100}\times\frac{\overset{33}{\cancel{66}}x}{\underset{2}{\cancel{50}}}\right)$$

$$= \text{Rs. } \left(\frac{66x}{50} + \frac{33x}{100}\right)$$

$$= \text{Rs. } \left(\frac{132x + 33x}{100}\right)$$

$$= \text{Rs. } \left(\frac{\overset{33}{\cancel{165}}x}{\underset{20}{\cancel{100}}}\right)$$

$$= \text{Rs. } \left(\frac{33x}{20}\right)$$

S.P. for the retailer = C.P. for the customer

$$= \text{Rs. } \left(\frac{33x}{20}\right)$$

According to the question,

$$\frac{33x}{20} = 6600$$

$$\Rightarrow x = \frac{\overset{200}{\cancel{6600}} \times 20}{\cancel{33}}$$

$$= 4000$$

$\therefore$ Cost price of the mobile = Rs. 4000.

Method - 2

Retailer sells the mobile to the customer at a gain of 25%

25% gain means if C.P. = Rs. 100, S.P = Rs. 125.

Retailer sells it for Rs. 6600

S.P.	C.P.
125	100
6600	?

$$\text{C.P.} = \text{Rs. } \left(\overset{4}{100} \times \frac{\overset{1320}{6600}}{\underset{5}{125}}\right)$$

$$= \text{Rs. } 5280$$

$\therefore$ For the retailer C.P. = Rs. 5280

$\therefore$ For the wholesaler S.P = Rs. 5280

But wholesaler gets 20% gain

i.e. if C.P = Rs.100 S.P = Rs. 120

S.P.	C.P.
120	100
5280	?

$$\text{C.P.} = \text{Rs.}\left(100\times\frac{\overset{44}{\cancel{5280}}}{\cancel{120}}\right)$$

$$= \text{Rs. } 4400$$

$\therefore$ C.P. for the wholesaler = Rs. 4400

$\therefore$ S.P. for the manufacturer = Rs. 4400

But manufacturer gets 10% gain

10% gain means if C.P = Rs. 100, S.P = Rs. 110

S.P.	C.P.
110	100
4400	?

$$\text{C.P.} = \text{Rs.}\left(100\times\frac{\overset{40}{\cancel{4400}}}{\cancel{110}}\right)$$

$$= \text{Rs. } 4000$$

$\therefore$ C.P. for the mobile = Rs. 4000.

Example 14: A difference of Rs. 4.50 in the sale price of an article would mean a gain of 7½% instead of a loss of 12½%. What was the cost price of the article?

Solution: Method-1

7½% gain means if C.P. = Rs. 100 then S.P = Rs. 107.50

12½% loss means if C.P = Rs. 100 then S.P = Rs. 87.50

Difference between the two sale prices = Rs. (107.50 – 87.50) = Rs. 20

Now if the difference between the two sale prices is Rs. 20 then cost price of the article is Rs. 100

Difference	C.P
20	100
4.5	?

$$\text{C.P} = \text{Rs.}\left(10\cancel{0}\times\frac{4.5}{2\cancel{0}}\right)$$

$$= \text{Rs. }\frac{45}{2} = \text{Rs. } 22.50$$

Method 2

Let cost price of the article = Rs. x

$$7\tfrac{1}{2}\%\text{ gain means S.P} = \text{Rs.}\left(x + 7\frac{1}{2}\%\text{ of } x\right)$$

$$= \text{Rs.}\left(x + \frac{\cancel{15}^{3}}{2\times\underset{20}{\cancel{100}}}x\right)$$

$$= \text{Rs. }\frac{43x}{40}$$

12½% loss means S.P = Rs. $\left(x-12\frac{1}{2}\%\text{of }x\right)$

$$= \text{Rs. } \left(x-\frac{\cancel{25}}{2\times\cancel{100}_{4}}\times x\right)$$

$$= \text{Rs. } \frac{7x}{8}$$

According to the question,

$$\frac{43x}{40}-\frac{7x}{8}=4.5 \Rightarrow \frac{43x-35x}{40}=4.5$$

$$\Rightarrow \frac{\cancel{8}x}{\cancel{40}_{5}}=4.5 \Rightarrow x=4.5\times 5=22.5$$

$\therefore$ C.P. = Rs. 22.50

Example 15 : If the price of an article is increased by 10%, 4 less articles can be bought for Rs. 440. Find the original price of an article.

Solution: Let Rs. x = the original price of an article

Number of articles bought for Rs. $x = 1$

$\therefore$ Number of articles bought for Rs. 440 = $\frac{440}{x}$

The price of an article is increased by 10%

$\therefore$ New price of an article = Rs. (x + 10% of x)

$$= \text{Rs. } \left(x+\frac{10}{100}x\right)$$

$$= \text{Rs. } \frac{11x}{10}$$

With the new price

Number of article bought for Rs. $\frac{11x}{10}=1$

$\therefore$ Number of article bought for Rs. 1 $=\frac{1}{\frac{11x}{10}}$

$\therefore$ Number of article bought for Rs. 440 $= \dfrac{440}{\frac{11x}{10}}$

$$= \overset{40}{\cancel{440}} \times \frac{10}{\cancel{11}x}$$

$$= \frac{400}{x}$$

According to the question,

$$\frac{440}{x} = \frac{400}{x} + 4$$

$$\Rightarrow \frac{440}{x} - \frac{400}{x} = 4$$

$$\Rightarrow \frac{440 - 400}{x} = 4$$

$$\Rightarrow x = \frac{40}{4} = 10$$

$\therefore$ Required original price is Rs. 10.

Example 16 : If the price of an article decreases by 20%, 6 more articles can be bought for Rs. 3000. What was the original price of each article?

Solution: Let the original price of each article = Rs. x.

Number of article bought for Rs. $x = 1$

$\therefore$ Number of article bought for Rs. 3000 = $\dfrac{3000}{x}$

Price of an article is decreased by 20%

$\therefore$ New price of an article = Rs. (x – 20% of x)

$$= \text{Rs. } \left(x - \frac{20}{100} \times x\right)$$

$$= \text{Rs. } \left(x - \frac{x}{5}\right)$$

$$= \text{Rs. } \frac{4x}{5}$$

With the new price

Number of article bought for Rs. $\frac{4x}{5} = 1$

Number of article bought for Rs. 3000 = $\frac{3000}{\frac{4x}{5}}$

$$= \overset{750}{\cancel{3000}} \times \frac{5}{\cancel{4}x}$$

$$= \frac{3750}{x}$$

According to the question,

$$\frac{3000}{x} = \frac{3750}{x} - 6$$

$$\Rightarrow \frac{3750}{x} = \frac{3000}{x} + 6$$

$$\Rightarrow 3750 - 3000 = 6x$$

$$\Rightarrow 750 = 6x \Rightarrow x = \frac{\overset{125}{\cancel{750}}}{\cancel{6}}$$

$\therefore$ Original price of each article = Rs. 125.

Example 17 : A man sells a watch at a profit of 20% on the selling price. If his cost price would have been 5% less and selling price Rs. 30 less, there would have been a profit of 25%. Find original cost price of the watch.

Solution: Let original C.P. of the article = Rs. x

20% profit on S.P. means if S.P = Rs. 100 then C.P. = Rs. 80

C.P.	S.P.
80	100
x	?

S.P. = Rs. $\left(100 \times \frac{x}{80}\right)$

$= \text{Rs.} \frac{5x}{4}$

Assumed cost price of the watch = 5% less

Assumed C.P. = Rs. x – 5% of x

$$= \text{Rs.} \left(x - \frac{\cancel{5}}{\underset{20}{\cancel{100}}} \times x\right)$$

$$= \text{Rs.} \frac{19\,x}{20}$$

Assumed S.P. = Rs. $\left(\frac{5x}{4}-30\right)$

and profit = 25% of assumed C.P.

$$= \text{Rs. } \left(\frac{\cancel{25}}{\underset{4}{\cancel{100}}}\times\frac{19x}{20}\right)$$

$$= \text{Rs. } \left(\frac{19x}{80}\right)$$

Now S.P. = C.P. + Profit

$$\therefore \frac{5x}{4}-30=\frac{19x}{20}+\frac{19x}{80}$$

$$=\frac{76x+19x}{80}$$

$$=\frac{{}^{19}\cancel{95}x}{\underset{16}{\cancel{80}}}=\frac{19x}{16}$$

$$\Rightarrow \frac{5x}{4}-\frac{19x}{16}=30$$

$$\Rightarrow 20x-19x=16\times 30$$

$$\Rightarrow x=480$$

$\therefore$ Original cost price of the watch = Rs. 480.

Example 18 : A man purchased 840 oranges. He sells $\frac{1}{4}$th of these at 20% less. At what percent profit should he sell the remaining oranges so as to make an overall profit at 20%?

Solution: Let cost price of all the 840 oranges = Rs. x

Overall profit is 20% means selling price of all the oranges = Rs. (x + 20% of x)

$$= \text{Rs. } \left(x+\frac{20}{100}\times x\right)$$

$$= \text{Rs. } \frac{6x}{5}$$

He sells $\frac{1}{4}$th of the oranges at 20% less

$\therefore$ S.P. of these $\frac{1}{4}$th of the oranges

$$= \text{Rs. } \left(\frac{x}{4} - 20\% \text{ of } \frac{x}{4}\right)$$

$$= \text{Rs. } \left(\frac{x}{4} - \frac{\cancel{2}\cancel{0}}{10\cancel{0}} \times \frac{x}{\underset{2}{\cancel{4}}}\right)$$

$$= \text{Rs. } \left(\frac{x}{4} - \frac{x}{20}\right)$$

$$= \text{Rs. } \left(\frac{5x - x}{20}\right) = \frac{x}{5}$$

Cost price of remaining oranges $= \left(x - \frac{x}{4}\right) = \frac{3x}{4}$

Let the remaining oranges be sold at y% profit to make an overall profit of 20%

$\therefore$ Selling price of remaining oranges

$$= \text{Rs. } \left(\frac{3x}{4} + y\% \text{ of } \frac{3x}{4}\right)$$

$$= \text{Rs. } \left(\frac{3x}{4} + \frac{y}{100} \times \frac{3x}{4}\right)$$

Now S.P. of $\frac{1}{4}$th oranges + S.P. of $\frac{3}{4}$th oranges

= S.P. of all the oranges

$$\Rightarrow \frac{x}{5} + \left(\frac{3x}{4} + \frac{3xy}{400}\right) = \frac{6x}{5}$$

$$\Rightarrow \frac{1}{5} + \left(\frac{3}{4} + \frac{3y}{400}\right) = \frac{6}{5}$$

$$\Rightarrow \frac{3y}{400} = \frac{6}{5} - \frac{1}{5} - \frac{3}{4}$$

$$= \left(\frac{6-1}{5} - \frac{3}{4}\right)$$

$$=1-\frac{3}{4}=\frac{1}{4}$$

$$\Rightarrow y=\frac{1}{4}\times\frac{400}{3}=\frac{100}{3}=33\frac{1}{3}$$

$\therefore$ Required profit percent = $33\frac{1}{3}\%$.

Example 19 : A dishonest shopkeeper sell his goods at cost price but he uses a weight of 800 gm for the 1 kg weight. Find the gain percent.

Solution: Let C.P. of 1 kg = Rs. x

$\therefore$ C.P. of 800 gm = Rs. $800x$

S.P. of 800 gm = C.P. of 1 kg

= C.P. of 1000 gm

= Rs. 1000 x

(Since the shopkeeper uses 800 gm instead of 1 kg)

$\therefore$ Profit = S.P. – C.P.

= Rs. $1000x$ – Rs. $800x$

= Rs. $200x$

$\therefore$ Gain percent = $\frac{\text{gain}}{\text{C.P}}\times 100\%$

$$=\frac{\text{Rs. }200x}{\text{Rs. }800x}\times 100\%$$

$$=\frac{1}{4}\times 100\% = 25\%$$

Ans : 25%

Example 20 : A shopkeeper sells the goods at 44% loss on cost price but uses 30% less weight. What is his percentage profit or loss?

Solution: Let total weight of goods = x kg and C.P. of 1 kg = Rs. y

$\therefore$ C.P. of x kg goods = Rs. xy

He sells (x – 30% of x) kg goods

$$=x-\frac{30x}{100}=\frac{7x}{10}\text{ kg}$$

C.P. of $\frac{7x}{10}$ kg = Rs. $\frac{7xy}{10}$

Since he uses 30% less weight. Though he sold $\frac{7x}{10}$ kgm but he took price of x kg)

Loss on C.P = 44%

$$SP = C.P - Loss$$

$$= xy - 44\% \text{ of } xy$$

$$= xy - \frac{44}{100} \times xy$$

$$= \left(\frac{100-44}{100}\right)xy = \frac{\overset{14}{\cancel{56}}}{\underset{25}{\cancel{100}}}xy = \frac{14xy}{25}$$

Actual loss = Actual C.P. – S.P.

$$= \frac{7xy}{10} - \frac{14xy}{25}$$

$$= \frac{35xy - 28xy}{50}$$

$$= \frac{7xy}{50}$$

[Because he sold $\frac{7x}{10}$ kg goods

$\therefore$ C.P. of $\frac{7x}{10}$ kgm = $\frac{7xy}{10}$]

$$\text{Loss percentage} = \frac{\text{Loss}}{\text{C.P}} \times 100\%$$

$$= \frac{\frac{7xy}{50}}{\frac{7xy}{10}} \times 100\%$$

$$= \frac{\cancel{7xy}}{\cancel{50}} \times \frac{1\cancel{0}}{\cancel{7xy}} \times \overset{20}{\cancel{100}}\%$$

$$= 20\%$$

Ans : 20% loss.

Example 21 : A grocer sells rice at a profit of 20% and uses a weight which is 25% less. Find his overall percentage gain.

Solution: Let total weight of rice = x kg and C.P. of 1 kg = Rs. y

$\therefore$ C.P. of x kg rice = Rs. xy

He sells (x – 25% of x) kg rice

$$= \left(x - \frac{25}{100} \times x\right) \text{ kg rice}$$

$$= \left(x - \frac{x}{4}\right) \text{ kg rice}$$

$$= \frac{3x}{4} \text{ kg rice}$$

C.P. of $\frac{3x}{4}$ kg rice = Rs. $\frac{3xy}{4}$

Profit = 20%

S.P. = C.P. + profit

$= xy + 20\%$ of xy

$$= xy + \frac{20}{100}xy$$

$$= \frac{6xy}{5}$$

He uses 25% less weight. He sold $\frac{3x}{4}$ kg but he took price of x kg rice

Now gain = S.P. – C.P. of $\frac{3x}{4}$ kg rice

$$= \frac{6xy}{5} - \frac{3xy}{4}$$

$$= 3xy\left(\frac{2}{5} - \frac{1}{4}\right)$$

$$= 3xy \times \left(\frac{8-5}{20}\right)$$

$$= 3xy \times \frac{3}{20} = \frac{9xy}{20}$$

$\therefore$ Overall gain percent $= \frac{\text{gain}}{\text{C.P}} \times 100\%$

$$= \frac{\frac{9xy}{20}}{\frac{3xy}{4}} \times 100\%$$

$$= \frac{\overset{3}{\cancel{9}}\cancel{xy}}{\cancel{2}\cancel{0}} \times \frac{\overset{2}{\cancel{4}}}{\cancel{3}\cancel{xy}} \times 10\cancel{0}\%$$

= 60% **Ans.** 60%

Exercise

1. If the cost price of an article is Rs. 800 and profit is 20%, find the selling price of the article.
2. A mechanic buys a bicycle for Rs. 1125 and spends Rs. 75 for its repairs. If he sells it for Rs. 1,500, find his profit or loss percent on selling price.
3. A man bought apples at the rate of 6 for Rs. 20 and sold them at 4 for Rs. 16. Find his percentage of profit.
4. If the cost price of 21 watches is equal to the selling price of 18 watches, what will be the gain percent in this transaction?
5. If the selling price of $\frac{2}{3}$rd of a certain quantity of milk be equal to the cost price of whole milk. What will be the gain percent in this transaction?
6. A shopkeeper sells the goods at 10% loss on cost price but uses 20% less weight. What is his percentage profit or loss?
7. Arun sells a watch to Biman at a gain of 20% and Biman sells it to Samir at a loss of 10%. If Samir pays Rs. 1080, how much did it cost to Arun?
8. By selling an article for Rs. 96, double the profit is obtained than the profit that would have been obtained by selling it for Rs. 84. What is the cost price of the article?
9. If selling price of an article is $\frac{4}{3}$ of its cost price. What is the profit percent?
10. A man sold a calculator at a loss of 5%. Had he purchased it at 10% less price and sold it at Rs. 126 more, then he would have gained $\frac{1}{4}$th of the C.P. Find the C.P. of the calculator.
11. Anil bought 80 chairs for Rs. 200 each. He sells 20 of them at a gain of 4%. At what gain percent should he sell the remainder so as to gain 12% on the whole?
12. If 5% of selling price of an article is equal to 6% of its cost price and 8% of selling price exceeds 9% of cost price by Rs. 4.50. Find the cost price and selling price of the article.
13. A shopkeeper sold a fan at Rs. 1230 and suffers a loss of 18%. What would have been the gain or loss percent of it had been sold for Rs. 1600?

Answers

1. Rs. 960 (2) 20% profit on S.P. (3) 20% (4) $16\frac{2}{3}\%$ (5) 50%

(6) 12½% gain (7) Rs. 1000 (8) Rs. 72 (9) $33\frac{1}{3}\%$ (10) Rs. 720

(11) $14\frac{2}{3}\%$ (12) C.P. = Rs. 750, S.P. = Rs. 900 (13) gain $6\frac{2}{3}\%$

Discount

Few Terms

Marked price: Manufacturers, Publishers or Businessmen generally mark their articles in a price higher than the actual selling price. The price is known as Marked Price (M.P.) or List Price (L.P.) or Catalogue Price.

Discount :

Any concession or reduction given on the marked price of an article to the buyer by the seller is known as Discount. In such situation, the selling price may be expressed as follows

$$\boxed{\text{Selling Price} = \text{Marked Price} - \text{Discount}}$$

Note:- Discount is always calculated on Marked Price.

Types of Discount:

In general, there are three types of discounts. These are

(i) Trade Discount: A rebate or allowance which is allowed on marked price is called Trade Discount. So, trade discount is a kind of discount given on the marked price of an article by allowing a margin of profit to the retailer.

(ii) Cash Discount: A concession allowed for prompt payment in cash is known as Cash Discount.

(iii) Retail Discount : It is a kind of discount given by the retail trader to the end consumer. It generally happens that the retail trader wants to dispose off their old stocks or clear off their off season stocks.

This retail discount is a kind of discount given by the retailer on the marked price of an article.

Note: Discount is calculated as percentage on M.P. Profit or loss is calculated as percentage on C.P. (unless otherwise stated). Discount connects M.P. and S.P. whereas profit or loss connects S.P. and C.P.

To find M.P. from C.P., at first we are to find S.P. from C.P. and then calculate M.P. from S.P.

Series of Discounts:

Sometimes a shopkeeper may offer a series of discounts for various reasons.

In such cases

(i) The first discount is calculated on the marked price.

(ii) The second discount is calculated on the reduced price obtained after giving the 1st discount.

(iii) The third discount is calculated on the reduced price obtained after giving the second discount, and so on.

Worked out Examples

Example 1 : A discount of 15% is allowed on the marked price Rs. 24. Find the selling price.

Solution: 15% discount on marked price means if M.P. = Rs. 100 then S.P = Rs. 85.

M.P.	S.P.
100	85
24	?

$$\text{Required S.P.} = \text{Rs.}\left(\overset{17}{\cancel{85}}\times\frac{\overset{6}{\cancel{24}}}{\underset{\cancel{25}\,5}{\cancel{100}}}\right)$$

$$= \text{Rs.}\left(\frac{17\times 6}{5}\right)$$

$$= \text{Rs.}\left(\frac{17\times 12}{10}\right)$$

$$= \text{Rs.}\ \frac{204}{10}$$

$$= \text{Rs. } 20.40$$

$\therefore$ Required selling price = Rs. 20.40.

Example 2 : After allowing a discount of 12% an article is sold at Rs. 211.20; find the marked price.

Solution: 12% discount means if M.P = Rs. 100 then S.P. = Rs. 88.

Let Rs. x be the M.P. when S.P. = Rs. 211.20

S.P.	M.P.
88	100
211.20	x

$$x = \text{Rs.}\left(100\times\frac{211.20}{88}\right)$$

$$= \text{Rs.}\left(10\times\frac{\overset{24\,\cancel{192}}{\cancel{2112}}}{\underset{\cancel{8}}{\cancel{88}}}\right)$$

$$= \text{Rs. } 240$$

$\therefore$ Required marked price of the article is Rs. 240.

Example 3 : A photographer allows a discount of 10% on the advertised price of a camera. What price must be marked on the camera which costs him Rs. 600 to make a profit of 20%?

Solution: C.P. of the article = Rs. 600

Profit = 20% on C.P.

$\therefore$ S.P. = C.P + Profit

= Rs. (600 + 20% of 600)

$$= \text{Rs.}\left(600+\frac{20}{1\cancel{00}}\times 6\cancel{00}\right) = \text{Rs. } (600 + 120)$$

$$= \text{Rs. } 720$$

∴ S.P. = Rs. 720

Now discount = 10%

∴ If M.P. = Rs. 100, S.P. = Rs. 90

S.P.	MP
90	100
720	?

$$\therefore \text{ M.P.} = \text{Rs.}\left(100\times\frac{\overset{8}{\cancel{720}}}{\cancel{90}}\right)$$

$$= \text{Rs. } 800$$

∴ Required marked price of the article is Rs. 800.

Example 4 : A dealer allows 12% trade discount and 5% cash discount. Find the S.P. if M.P. of an article is Rs. 200.

Solution: M.P. of the article = Rs. 200

12% trade discount

If M.P. = Rs. 100, S.P. = Rs. 88

S.P.	M.P.
88	100
?	200

$$\text{S.P.} = \text{Rs.}\left(88\times\frac{2\cancel{00}}{1\cancel{00}}\right)$$

$$= \text{Rs. } 176$$

5% cash discount.

5% discount will be calculated on the reduced value i.e. on Rs. 176.

5% discount means if M.P. = Rs. 100 then S.P. = Rs. 95.

M.P.	S.P.
100	95
176	?

$$\text{S.P.} = \text{Rs.}\left(\overset{19}{\cancel{95}}\times\frac{\overset{88}{\cancel{176}}}{\underset{\underset{10}{\cancel{20}}}{\cancel{100}}}\right)$$

$$= \text{Rs. }\frac{1672}{10} = \text{Rs. } 167.2$$

Required S.P. = Rs. 167.20.

Note (i) : You may `find the S.P. as follows

S.P.= Rs. (176 – 5% of 176)

$$= \text{Rs.} \left(176 - \frac{\cancel{8}}{\underset{20}{\cancel{100}}} \times \overset{8.8}{\cancel{176}}\right)$$

$$= \text{Rs.}\ (176 - 8.8) = \text{Rs.}\ 167.20.$$

(ii) When a trader allows both trade and cash discounts, then the trade discount should be first calculated followed by the cash discount on the reduced value.

Example 5 : A trader allows a discount of 20%, 10% and 10% on the marked price. Find the S.P. of an article marked at Rs. 220.

Solution: Marked price = Rs. 220.

1st discount = 20%

$\therefore$ 1st reduced price = Rs. (220 – 20% of 220)

$$= \text{Rs.} \left(220 - \frac{2\cancel{0}}{1\cancel{0}\cancel{0}} \times 22\cancel{0}\right)$$

$$= \text{Rs.}\ (220 - 44)$$

$$= \text{Rs.}\ 176$$

2nd discount = 10%

$\therefore$ 2nd reduced price= Rs. (176 – 10% of 176)

$$= \text{Rs.} \left(176 - \frac{1\cancel{0}}{10\cancel{0}} \times 176\right)$$

$$= \text{Rs.}\ (176 - 17.6)$$

$$= \text{Rs.}\ 158.40$$

3rd discount = 10%

$\therefore$ 3rd reduced price = Rs. (158.40 – 10% of 158.40)

$$= \text{Rs.} \left(158.40 - \frac{1\cancel{0}}{10\cancel{0}} \times 158.4\right)$$

$$= \text{Rs.}\ (158.40 - 15.84)$$

$$= \text{Rs.}\ 142.56$$

Ans: 142.56.

$\therefore$ Required S.P. = Rs. 142.56.

Example 6 : Having marked his article 25% above cost, a trader makes a profit of Rs. 125 after allowing a discount of 15%, find the C.P.

Solution: Method-1

Let M.P. = Rs. 100

Discount = 15%, $\therefore$ M.P. = Rs. 100, S.P. = Rs. 85.

Also M.P. is 25% above cost price

i.e. if C.P. = Rs. 100, M.P. = Rs. 125

M.P.	C.P.
125	100
100	?

$$\text{C.P.} = \text{Rs.}\left(\overset{4}{\cancel{100}}\times\frac{\overset{20}{\cancel{100}}}{\underset{5}{\cancel{125}}}\right)$$

$$= \text{Rs. } 80$$

Thus if M.P. = Rs. 100 then S.P. = Rs. 85 and C.P. = Rs. 80

Thus when M.P. = Rs. 100 then profit = S.P. – CP

= Rs. 85 – Rs. 80

= Rs. 5

Thus if profit = Rs. 5 then C.P. = Rs. 80

Profit	C.P.
5	80
125	?

$$\text{C.P.} = \text{Rs.}\left(80\times\frac{\overset{25}{\cancel{125}}}{\cancel{5}}\right) = \text{Rs. } 2000$$

$\therefore$ Required C.P. = Rs. 2000.

Method - 2

Let C.P. = Rs. x

M.P. is 25% as above C.P.

M.P. = Rs. (x + 25% of x)

$$= \text{Rs.}\left(x+\frac{25}{100}\times x\right)$$

$$= \text{Rs.}\left(x+\frac{x}{4}\right) = \text{Rs. }\frac{5x}{4}$$

Discount = 15%

$$\therefore \text{ S.P.} = \text{Rs.}\left(\frac{5x}{4} - 15\% \text{ of } \frac{5x}{4}\right)$$

$$= \text{Rs.}\left(\frac{5x}{4} - \frac{\overset{3}{\cancel{15}}}{\underset{\cancel{20}\ 4}{\cancel{100}}}\times\frac{\cancel{5}x}{4}\right)$$

$$= \text{Rs. } \left(\frac{5x}{4} - \frac{3x}{16}\right)$$

$$= \text{Rs. } \left(\frac{20x - 3x}{16}\right) = \text{Rs. } \frac{17x}{16}$$

Profit = S.P. – C.P.

$$= \text{Rs. } \frac{17x}{16} - \text{Rs. } x$$

$$= \text{Rs. } \left(\frac{17x}{16} - x\right)$$

$$= \text{Rs. } \frac{x}{16}$$

According to the question,

$$\frac{x}{16} = 125 \Rightarrow x = 16 \times 125$$

$$= 2000$$

$\therefore$ Required C.P. = Rs. 2000.

Example 7 : The list price of an article is 25% above the S.P. and C.P. is 40% below the list price. Find the rate of discount and profit.

Solution: Let S.P. = Rs. 100

Given that the list price is 25% above the S.P.

$\therefore$ List price = Rs. (100 + 25% of 100)

$$= \text{Rs. } \left(100 + \frac{25}{100} \times 100\right)$$

$$= \text{Rs. } 125.$$

Again C.P. is 40% below the list price

$\therefore$ C.P. = Rs. (125 – 40% of 125)

$$= \text{Rs. } \left(125 - \frac{\overset{2}{\cancel{40}}}{\underset{5}{\cancel{100}}} \times \overset{25}{\cancel{125}}\right)$$

$$= \text{Rs. } (125 - 50)$$

$$= \text{Rs. } 75$$

Amount of discount = L.P. – S.P.

= Rs. (125 – 100)

= Rs. 25

Rate of discount is the discount when the L.P. = Rs. 100.

List price	Discount
125	25
100	?

$$? = \text{Rs.}\left(\cancel{25}\times\frac{\overset{20}{\cancel{100}}}{\underset{\cancel{5}}{\cancel{125}}}\right)$$

= Rs. 20

∴ Rate of discount = 20%

Now amount of profit = S.P. – C.P

= Rs. (100 – 75)

= Rs.25

C.P.	Profit
75	25
100	x

$$x = \text{Rs.}\left(\cancel{25}\times\frac{100}{\underset{3}{\cancel{75}}}\right)$$

$$= \text{Rs.}\ \frac{100}{3} = \text{Rs.}\ 33\frac{1}{3}$$

Thus when C.P. = Rs. 100 then profit = Rs. $33\frac{1}{3}$

∴ Percentage of profit = $33\frac{1}{3}\%$.

Example 8 : A bicycle agent allows 25% discount on his advertised prices and then makes a profit of 20% on his outlay. What is the advertised price of a machine on which he gains Rs. 30?

Solution: Advertised price means marked price

Let M.P. = Rs. 100

25% discount ∴ S.P. = Rs. (100 – 25% of 100)

= Rs. (100 – 25)

= Rs. 75

Profit = 20%, i.e. if C.P = Rs. 100 then S.P. = Rs. 120 and profit = Rs. 20.

S.P.	C.P.
120	100
75	x

$$x = \text{Rs.}\left(\overset{5}{\cancel{100}}\times\frac{\overset{25}{\cancel{75}}}{\underset{\cancel{4}\,2}{\cancel{120}}}\right)$$

$$= \text{Rs.}\ \frac{125}{2}$$

Thus if M.P. = Rs. 100, then

$$\text{S.P.} = \text{Rs. } 75 \text{ and C.P.} = \text{Rs. } \frac{125}{2}$$

$$\therefore \text{ Profit } = \text{Rs. } \left(75 - \frac{125}{2}\right)$$

$$= \text{Rs. } \left(\frac{150 - 125}{2}\right)$$

$$= \text{Rs. } \frac{25}{2}$$

Profit	M.P.
$\frac{25}{2}$	100
30	x

$$x = \text{Rs. } \left(100 \times \frac{30}{\frac{25}{2}}\right)$$

$$= \text{Rs. } \left(\overset{4}{\cancel{100}} \times 30 \times \frac{2}{\cancel{25}}\right)$$

$$= \text{Rs. } 240$$

$\therefore$ Required advertised price of the machine is Rs. 240.

Example 9 : A scooter dealer bought an old scooter listed at Rs. 26,000 and got successive discounts of 5% and 10%. He spends some amount on repairs and sells it for Rs. 27,500 thereby gains 10%. Find the amount spent on repairs.

Solution: Here M.P. = Rs. 26,000

1st discount = 5%

$$\text{1st reduced price} = \text{Rs. } (26{,}000 - 5\% \text{ of } 26{,}000)$$

$$= \text{Rs. } \left(26{,}000 - \frac{5}{1\cancel{0}\cancel{0}} \times 26{,}0\cancel{0}\cancel{0}\right)$$

$$= \text{Rs. } (26{,}000 - 1{,}300)$$

$$= \text{Rs. } 24{,}700$$

2nd discount = 10%

$$\text{2nd reduced price} = \text{Rs. } (24{,}700 - 10\% \text{ of } 24{,}700)$$

$$= \text{Rs. } \left(24{,}700 - \frac{1\cancel{0}}{1\cancel{0}\cancel{0}} \times 24{,}70\cancel{0}\right)$$

$$= \text{Rs. } (24{,}700 - 2{,}470)$$

$$= \text{Rs. } 22{,}230$$

Now gain = 10%

$\therefore$ If C.P. = Rs. 100 then S.P. = Rs. 110

Given that S.P. = Rs. 27,500

Let C.P. of the scooter including the repairing cost = Rs. x

S.P.	C.P.
110	100
27,500	x

$$x = \text{Rs.}\left(100 \times \frac{\overset{250}{\cancel{27{,}500}}}{\cancel{110}}\right)$$

$$= \text{Rs. } 25{,}000$$

$$\therefore \text{ C.P.} = \text{Rs. } 25{,}000$$

$\therefore$ Amount spent on repairs + 2nd reduced price = C.P.

$\therefore$ Amount spent on repairs = C.P. – 2nd reduced price

$$= \text{Rs. } (25{,}000 - 22{,}230)$$

$$= \text{Rs. } 2770$$

Ans: Rs. 2770.

Example 10 : A trader allows his customer 5% discount on his list price. What would be the list price of an article costing Rs. 712.50 to make a profit of $33\frac{1}{3}\%$?

Solution: Let list price = Rs. 100

$\therefore$ S.P. = Rs. (100 – 5% of 100)

$$= \text{Rs. } \left(100 - \frac{5}{100} \times 100\right)$$

$$= \text{Rs. } 95$$

Profit = $33\frac{1}{3}\% = \frac{100}{3}\%$

If C.P. = Rs. 100 then S.P. = Rs. $\left(100 + \frac{100}{3}\right)$ = Rs. $\frac{400}{3}$

S.P.	C.P.
$\frac{400}{3}$	100
95	x

$$x = \text{Rs.}\left(100 \times \frac{95}{\frac{400}{3}}\right)$$

$$= \text{Rs. } \left(1\cancel{00} \times 95 \times \frac{3}{4\cancel{00}}\right)$$

$$= \text{Rs. } \frac{285}{4}$$

Thus if M.P. = Rs. 100 then S.P. = Rs. 95 and C.P. = Rs. $\frac{285}{4}$

C.P.	M.P.
$\frac{285}{4}$	100
712.5	y

$$y = \text{Rs.}\left(100 \times \frac{712.5}{\frac{285}{4}}\right)$$

$$= \text{Rs.}\left(10\cancel{0} \times \frac{\overset{25\ \cancel{475}\ \cancel{1425}}{\cancel{7125}}}{\underset{\cancel{57}\ \cancel{19}}{\cancel{285}\cancel{0}}} \times 4\right) = \text{Rs. } (25 \times 10 \times 4)$$

$$= \text{Rs. } 1000$$

∴ Required list price of the article is Rs. 1000.

Example 11 : A watch trader allows 20% trade discount and 20% cash discount. What would be the list price of a watch costing Rs. 140 so as to make profit of 30% on selling price?

Solution: Let list price = Rs. 100

20% trade discount

1st discount = 20%

1st reduced price = Rs. (100 – 20% of 100)

= Rs. (100 – 20)

= Rs. 80

20% cash discount

2nd discount = 20%

2nd reduced price = Rs. (80 – 20% of 80)

$$= \text{Rs.}\left(80 - \frac{2\cancel{0}}{1\cancel{0}\cancel{0}} \times 8\cancel{0}\right)$$

= Rs. (80 – 16)

= Rs. 64

∴ S.P. of the article = Rs. 64 (if M.P = Rs. 100)

30% profit on S.P. means

if S.P = Rs. 100 then C.P. = Rs. 70

S.P.	C.P.
100	70
64	x

$$x = \text{Rs.}\left(\frac{7\cancel{0} \times 64}{10\cancel{0}}\right)$$

$$= \frac{448}{10}$$

= Rs. 44.8

CP is Rs. 44.8 then list price = Rs. 100

$\therefore$ CP is Rs. 140 then list price is

$$= \text{Rs.}\left(\overset{25}{\cancel{100}}\times\overset{\overset{5}{\cancel{10}}}{\cancel{140}}\times\frac{\overset{5}{\cancel{10}}}{\underset{\underset{\underset{\underset{2}{\cancel{4}}}{\cancel{8}}}{\cancel{32}}}{\cancel{448}}}\right)$$

$$= \text{Rs.}\left(\frac{625}{2}\right)$$

$$= \text{Rs. } 312,50$$

$\therefore$ Required list price of the article is Rs. 312.50.

Example 12 : An article with a marked price of Rs. 625 is sold at a discount of 20% by making a profit of 25%. Find the cost price of the article.

Solution: Let M.P = Rs. 100

20% discount means if M.P = Rs. 100 then S.P. = Rs. (100 – 20% of 100) = Rs. 80

25% profit means if C.P. = Rs. 100 then S.P. = Rs. 125

S.P.	C.P.
125	100
80	x

$$x = \text{Rs.}\left(\overset{4}{\cancel{100}}\times\frac{\overset{15}{\cancel{80}}}{\underset{\cancel{5}}{\cancel{125}}}\right)$$

$$= \text{Rs. } 64$$

$\therefore$ C.P. = Rs. 64

Thus if M.P. = Rs. 100 then C.P. = Rs. 64

M.P.	C.P.
100	64
625	y

$$y = \text{Rs.}\left(\overset{16}{\cancel{64}}\times\frac{\overset{25}{\cancel{625}}}{\underset{\cancel{4}}{\cancel{100}}}\right)$$

= Rs. 400 $\therefore$ Required C.P. = Rs. 400.

Exercise

1. A discount of 10% is allowed on the marked price Rs. 500, find the selling price.
2. A dealer allows 15% trade discount and 10% cash discount. Find the S.P. if the M.P. of an article is Rs. 400.
3. A trader allows a discount of 20%, 10% and 10% on the marked price. Find the sale price of an article marked at Rs. 345.
4. A bicycle agent allows 20% discount on his marked price and makes 20% profit on his outlay. What is the marked price of the bicycle on which he gains Rs. 240?
5. Having marked his article 25% above cost, a trader makes a profit of Rs. 300 after allowing a discount of 10%, find the cost price.
6. A dealer altered his trade discount from 15% to 12%. Find the percentage of selling prices altered.
7. An article was sold at a loss of 10%. If it was sold for Rs. 50 more, there would have been a profit of 15%. At what price should it be sold to gain $6\frac{1}{4}\%$?
8. A person buys an article and sells it at a profit of 5%. If he had bought it at 5% less and sold it for Re 0.37 less he would have gained 10%; find the original cost price.
9. A dealer is offered a discount of $14\frac{1}{4}\%$ by one wholesaler and 10% and 5% by another. Find which is cheaper. If he purchases at cheaper rate and sells at 7% below the list price, find the profit he makes.
10. A trader is offered a discount of 15% by one wholesaler, while he is offered a discount of 12% and further cash discount of 3% by another. Find which is cheaper. If he purchases at the cheaper rate and sells at 5% below list, what rate percent profit does he earn (i) on his C.P. (ii) On his S.P.
11. How much percent above the cost price should a shopkeeper mark his goods so that after allowing a discount of 20% on the marked price, he gains 25%?
12. Which of the following discount series is better to the customer
    (i) 20%, 15% and 10%
    (ii) 25%, 12% and 8%

Answers

1. Rs. 450 2. Rs. 306 3. Rs. 223.56
4. Rs. 1800 5. Rs. 2,400 6. 3.53%
7. Rs. 180, 2nd part = Rs. 212.50 8. Rs. 74
9. 2nd is cheaper and profit = 8.77%
10. 1st offer (i) 11.76% (ii) 10.53%
11. 56.25% 12. 2nd

Mixture

Introduction:

Mixture or Alligation is a rule to find

(a) The ratio in which two or more ingredients at their respective prices should be mixed to give a mixture at a given price.

(b) The mean or average price of a mixture when the prices of two or more ingredients which may be mixed together and the proportion in which they are mixed are given

Here, cost price of a unit quantity of mixture is called the mean price.

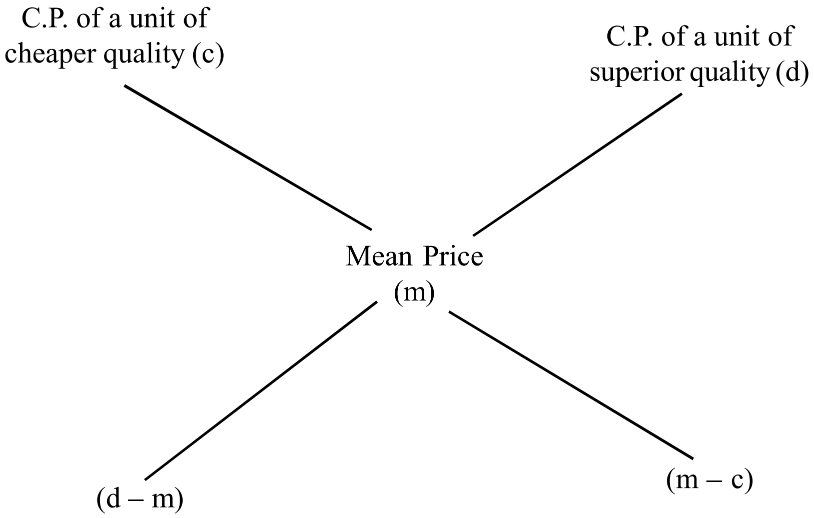
Alligation Rule :

Suppose, Rs. d per unit be the price of first ingredient (superior quality) mixed with another ingredient cheaper quality of price Rs. c per unit to form a mixture whose mean price is Rs. m per unit, then the two ingredients must be mixed in the ratio

$$\frac{\text{Quantity of cheaper}}{\text{Quality of superior}} = \frac{\text{C.P (superior)} - \text{Mean price}}{\text{Mean price} - \text{C.P (cheaper)}}$$

i.e. the two ingredients are to be mixed in the inverse ratio of differences of their prices and the mean price.

The above rule may be represented as follows :



$$\therefore \frac{\text{Quantity of cheaper}}{\text{Quantity of superior quality}} = \frac{d-m}{m-c}$$

Proof : Let x kg of cheaper quality is mixed with y kg of superior quality.

$\therefore$ Price of cheaper ingredient = Rs. cx

Price of superior ingredient = Rs. dy

$\therefore$ Price of mixture = Rs. $(cx + dy)$ and quantity of mixture = $(x + y)$ kg

$\therefore$ Price of mixture per kgm = Rs. $\left(\frac{cx+dy}{x+y}\right)$

$$\therefore \frac{cx+dy}{x+y} = m$$

$$\Rightarrow cx + dy = mx + my$$

$$\Rightarrow mx - cx = dy - my$$

$$\Rightarrow (m-c)x = (d-m)y$$

$$\Rightarrow \frac{x}{y} = \frac{d-m}{m-c}$$

Working Rule

(i) All the prices must be expressed in the same unit.

(ii) The cost price (C.P.) of the mixture must lie in between the cost prices of the other ingredients.

(iii) All prices used should be cost prices. If in some cases, the selling price of the mixture at profit or loss is given, then find the corresponding cost price.

(iv) Now apply the following formula to find the required ratio

$$\frac{\text{Quantity of cheaper quality}}{\text{Quantity of superior quality}} = \frac{d-m}{m-c}$$

i.e. $\frac{\text{1st kind}}{\text{2nd kind}} = \frac{\text{difference of C.P. of mixture with the 2nd kind}}{\text{difference of C.P. of mixture with that of 1st kind}}$

Worked out Examples

Example 1 : In what ratio two varieties of tea one costing Rs. 27 per kg and the other costing Rs. 32 per kg should be blended to produce a blended variety of tea worth Rs. 30 per kg. How much should be the quantity of second variety of tea, if the first variety is 60 kg?

Solution :

C.P. of 1 kg of cheaper tea (Rs. 27)

C.P. of 1 kg of superior tea (Rs. 32)

Mean Price Rs. 30

Rs. 32 – Rs. 30 = Rs. 2

Rs. 30 – Rs. 27 = Rs. 3

The required ratio of the two varieties of tea is 2 : 3

i.e. $\frac{\text{Quantity of cheaper tea}}{\text{Quantity of superior tea}} = \frac{2}{3}$

$\Rightarrow \frac{60\text{ kg}}{\text{Quantity of superior tea}} = \frac{2}{3}$

$\Rightarrow$ Quantity of superior tea = $\left(\overset{30}{\cancel{60}} \times \frac{3}{\cancel{2}}\right)\text{kg} = 90\text{ kg}$

$\therefore$ The second variety of tea = 90 kg.

Example 2 : Sugar at Rs. 15 per kg is mixed with sugar at Rs. 20 per kg in the ratio 2 : 3. Find the price per kg of the mixture.

Solution : Let the price per kg of the mixture = Rs. x

i.e. mean price of the mixture = Rs. x

C.P. of 1 kg of cheaper sugar (Rs. 15)

C.P. of 1 kg of superior sigar (Rs. 20)

Mean Price Rs. x

Rs. $(20 - x)$

Rs. $(x - 15)$

$$\frac{\text{Quantity of cheaper sugar}}{\text{Quantity of superior sugar}} = \frac{20-x}{x-15}$$

According to the question,

$$\frac{20-x}{x-15} = \frac{2}{3}$$

$$\Rightarrow 60-3x = 2x-30$$

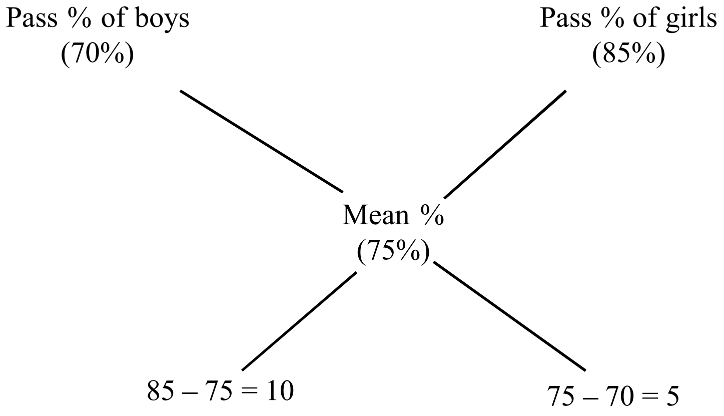
$$\Rightarrow 3x+2x = 60+30$$

$$\Rightarrow 5x = 90 \Rightarrow x = \frac{\cancel{90}^{18}}{\cancel{5}} = 18$$

Thus, the price per kg of the mixture = Rs. 18.

Example 3 : In an examination out of 480 students 85% of the girls and 70% of the boys passed. How many boys appeared in the examination if total pass percentage was 75% ?

Solution :



$$\frac{\text{Number of boys}}{\text{Number of girls}} = \frac{10}{5} = \frac{2}{1}$$

$$\Rightarrow \frac{\text{Number of girls}}{\text{Number of boys}} = \frac{1}{2}$$

$$\Rightarrow \frac{\text{Number of girls}}{\text{Number of boys}} + 1 = \frac{1}{2} + 1$$

$$\Rightarrow \frac{\text{Number of girls} + \text{Number of boys}}{\text{Number of boys}} = \frac{1+2}{2}$$

$$\Rightarrow \frac{\text{Number of students}}{\text{Number of boys}} = \frac{3}{2}$$

$$\Rightarrow \frac{480}{\text{Number of boys}} = \frac{3}{2}$$

$$\Rightarrow \text{Number of boys} = \overset{160}{\cancel{480}} \times \frac{2}{\cancel{3}}$$

$$= 320$$

$\therefore$ Number of boys appeared = 320.

Example 4 : In what ratio must water be added to spirit to gain 10% by selling it at the cost price?

Solution: Let the C.P. of spirit = Rs. 10 per litre

A/Q, S.P. of the mixture = Rs. 10 per litre

Gain = 10%

$\therefore$ If C.P. = Rs. 100 then S.P. = Rs. 110

S.P. of the mixture	C.P. of the mixture
110	100
10	x

$$\therefore\ x = \text{Rs.}\left(100 \times \frac{1\cancel{0}}{11\cancel{0}}\right) = \text{Rs.}\left(\frac{100}{11}\right)$$

$$\therefore\ \text{C.P. of the mixture} = \frac{100}{11} \text{ per litre.}$$

C.P. of water per litre: Re 0

C.P. of pure spirit per litre: Rs.10

Mean Price

Rs. $\frac{100}{11}$

Rs. $\left(10 - \frac{100}{11}\right)$

Rs. $\left(\frac{110-100}{11}\right)$

Rs. $\frac{10}{11}$

Rs. $\left(\frac{100}{11} - 0\right)$

Rs. $\left(\frac{100}{11}\right)$

$$\therefore \quad \frac{\text{Quantity of water}}{\text{Quantity of spirit}} = \frac{\frac{10}{11}}{\frac{100}{11}}$$

$$= \frac{10}{11} \times \frac{11}{100}$$

$$= \frac{1}{10}$$

$\therefore$ Quantity of water : Quantity of spirit

= 1 : 10.

Example 5 : In mixing two types of tea, 2% is wasted. In what ratio tea costing Rs. 60 per kg be mixed with tea costing Rs. 45 per kg so that by selling the mixture at Rs. 62.50 per kg, there is a gain of 25% on total outlay.

Solution: Cost price of 1st kind tea per kg = Rs. 60

Cost price of 2nd kind tea per kg = Rs. 45

Selling price of the mixture per kg = Rs. 62.50

Since 2% tea is wasted at the time of mixing, therefore if the trader buys 100 kg of tea, he can sell 98 kg of tea only

Now S.P. of the mixture = Rs. 62.50 per kg

$\therefore$ Total S.P. of the mixture = Rs. (62.50×98)

= Rs. 6,125

Now gain = 25%

$\therefore$ If C.P. = Rs. 100, then S.P. = Rs. 125

S.P.	C.P.
125	100
6125	x

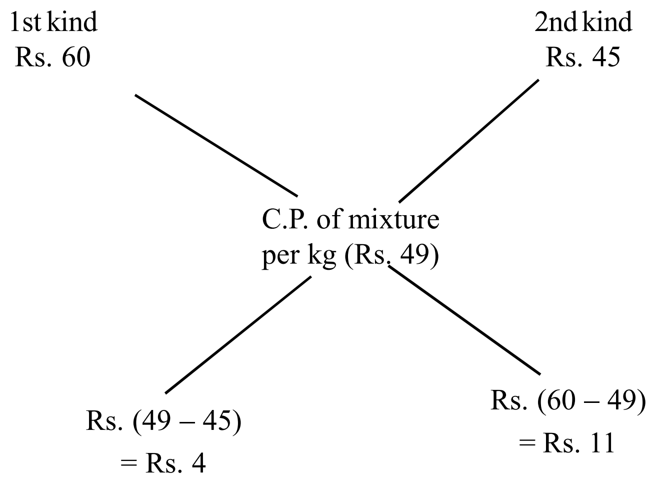
$$x = \text{Rs.}\ 100 \times \frac{\overset{\cancel{245}\ 49}{\cancel{6125}}}{\underset{\cancel{5}}{\cancel{125}}}$$

= Rs. 4,900

$\therefore$ Cost price of 100 kg mixture = Rs. 4,900

$$\therefore \text{ C.P. per kg mixture} = \text{Rs. } \frac{4900}{100}$$

= Rs. 49



∴ 1st kind : 2nd kind = 4 : 11

Ans : 4 : 11

Example 6 : A fruit seller buys oranges of two kinds. One kind of Rs. 24 per dozen and other at Rs. 16 per dozen. These were mixed up and he sells them for Rs. 24 per 15 and thereby makes 5% profit on his outlay. Find the mixing ratio of two kind of oranges.

Solution:

C.P. of 1st kind per dozen = Rs. 24

C.P. of 2nd kind per dozen = Rs. 16

S.P. of the mixture = Rs. 24 per 15

∴ S.P. of other mixture per dozen

$$= \text{Rs. } \left(\frac{24}{\underset{5}{\cancel{15}}} \times \overset{4}{\cancel{12}} \right)$$

$$= \text{Rs. } \frac{96}{5}$$

Profit = 5%

∴ C.P = Rs. 100 S.P. = Rs. 105

S.P.	C.P.
105	100
$\frac{96}{5}$	x

$$x = \text{Rs. } \left(100 \times \frac{96/5}{105} \right)$$

$$= \text{Rs.}\left(\overset{4}{\cancel{\overset{20}{\cancel{100}}}}\times\frac{\cancel{96}^{32}}{\cancel{5}}\times\frac{1}{\underset{\cancel{21}\,7}{\cancel{105}}}\right)$$

$$= \text{Rs.}\left(\frac{4\times32}{7}\right)$$

$$= \text{Rs.}\ \frac{128}{7}$$

1st kind
Rs. 24
(per dozen)

2nd kind
Rs. 16 (per dozen)

C.P. of mixture
per dozen Rs.
$\left(\frac{128}{7}\right)$

$$\text{Rs.}\left(\frac{128}{7}-16\right)$$

$$= \text{Rs.}\left(\frac{128-112}{7}\right)$$

$$= \text{Rs.}\ \frac{16}{7}$$

$$\text{Rs.}\left(24-\frac{128}{7}\right)$$

$$= \text{Rs.}\left(\frac{168-128}{7}\right)$$

$$= \text{Rs.}\ \frac{40}{7}$$

$$\text{1st kind : 2nd kind} = \frac{16}{7}:\frac{40}{7} = \frac{\overset{2}{\cancel{16}}}{\cancel{7}}\times\frac{\cancel{7}}{\underset{5}{\cancel{40}}} = \frac{2}{5} = 2:5$$

∴ Required mixing ratio = 2 : 5.

Example 7 : The ratio of milk and honey in two vessels are 2 : 3 and 4 : 3 respectively. At what ratio the mixtures from the two vessels must be mixed so that the new mixture may contain equal quantities of milk and honey.

Solution:

In the 1st vessel, Milk : Honey = 2 : 3

In the 2nd vessel, milk : honey = 4 : 3

Let total mixture in the 1st vessel = x litre

& total mixture in the 2nd vessel = y litre

$\therefore$ Amount of milk in the 1st vessel = $\frac{2x}{5}$ litre

Amount of money in the 1st vessel = $\frac{3x}{5}$ litre

Amount of milk in the 2nd vessel = $\frac{4y}{7}$ litre

Amount of honey in the 2nd vessel = $\frac{3y}{7}$ litre

Total amount of milk = $\left(\frac{2x}{5}+\frac{4y}{7}\right)$ litre

Total amount of honey = $\left(\frac{3x}{5}+\frac{3y}{7}\right)$ litre

According to the question,

$$\frac{2x}{5}+\frac{4y}{7}=\frac{3x}{5}+\frac{3y}{7}$$

$$\Rightarrow \frac{3x}{5}-\frac{2x}{5}=\frac{4y}{7}-\frac{3y}{7}$$

$$\Rightarrow \frac{x}{5}=\frac{y}{7} \Rightarrow \frac{x}{y}=\frac{5}{7}$$

$$\Rightarrow x:y=5:7$$

$\therefore$ The two given mixtures should be mixed in the ratio 5 : 7.

Example 8 : 30 gallons of mixture of spirit and water contains 60% spirit. How much water must be added to it to raise the amount of water to 75% ?

Solution: In 30 gallons of mixture of spirit and water contains 60% spirit.

$\therefore$ Amount of spirit = (60% of 30) gallons

$$= \frac{6\cancel{0}}{10\cancel{0}} \times 3\cancel{0} \text{ gallons}$$

= 18 gallons

$\therefore$ Amount of water = (30 – 18) gallons = 12 gallons

Let water added in the mixture = x gallons

$\therefore$ Amount of water in the new mixture = $(12 + x)$ gallons

Amount in new mixture = $(30 + x)$ gallons

Water in the new mixture = 75%

$\therefore$ 75% of $(30 + x) = 12 + x$

$$\Rightarrow \frac{75}{100} \times (30 + x) = 12 + x$$

$$\Rightarrow \frac{3}{4} \times (30 + x) = 12 + x$$

$$\Rightarrow 90 + 3x = 48 + 4x$$

$$\Rightarrow 4x - 3x = 90 - 48$$

$$\Rightarrow x = 42$$

$\therefore$ Water must be added = 42 gallons.

Ans: 42 gallons.

Example 9 : 42 litres of spirit contains paint and thinner in the ratio 4 : 3. How much thinner must be added to it in order that the mixture may contain paint and thinner in the ratio 4 : 5.

Solution:

Amount of paint in the mixture = $\left(\frac{4}{\cancel{7}} \times \overset{6}{\cancel{42}}\right)$ litres = 24 litres

Amount of thinner in the mixture= $\left(\frac{3}{\cancel{7}} \times \overset{6}{\cancel{42}}\right)$ litres = 18 litres

Let x litres of thinner must be added to the mixture of spirit to make the ratio of paint and thinner 4 : 5.

$\therefore \quad 24 : 18 + x = 4 : 5$

$$\Rightarrow \frac{24}{18 + x} = \frac{4}{5}$$

$$\Rightarrow (18 + x) \times 4 = 24 \times 5$$

$$\Rightarrow 18 \times 4 + 4x = 120$$

$$\Rightarrow 4x = 120 - 72 = 48$$

$\Rightarrow x = \frac{48}{4} = 12$ **Ans:** 12 litres

Example 10 : A milkman mixes 10 litres of milk costing Rs. 11 per litre with 20 litres of milk costing Rs. 14.50 per litre. He adds water to the mixture and makes a profit of 20% by selling Rs. 8 per litre. How much water does he mix?

Solution:

Total cost price of the mixture

$= \text{Rs. } (10 \times 11 + 20 \times 14.50)$

= Rs. (110 + 290)

= Rs. 400

Profit = 20%

$\therefore$ S.P. = Rs. (400 + 20% of 400)

$$= \text{Rs. } \left(400 + \frac{20}{1\cancel{00}} \times 4\cancel{00}\right)$$

= Rs. (400 + 80) = Rs. 480

He sells the mixture at Rs. 8 per litre

$\therefore$ Quantity of mixture sold $= \frac{480}{8}$ litre

= 60 litre

He bought milk = (10 + 20) litres

= 30 litres

$\therefore$ Quantity of water he added

= (60 – 30) litres

= 30 litres

Ans: 30 litres.

Example 11 : Three equal jars are filled with mixtures of spirit and water in the ratio 5 : 3, 3 : 1 and 9 : 7 respectively. The contents of three jars all emptied into a single jar. Find the ratio of spirit and water in the big jar.

Solution: Let capacity of each jar = x litre

In the 1st jar, spirit : water = 5 : 3

In the 2nd jar, spirit : water = 3 : 1

In the 3rd jar, spirit : water = 9 : 7

Amount of spirit in the 1st jar = $\frac{5x}{8}$

Amount of water in the 1st jar = $\frac{3x}{8}$

Amount of spirit in the 2nd jar = $\frac{3x}{4}$

Amount of water in the 2nd jar = $\frac{x}{4}$

Amount of spirit in the 3rd jar = $\frac{9x}{16}$

Amount of water in the 3rd jar = $\frac{7x}{16}$

Total amount of spirit $= \frac{5x}{8}+\frac{3x}{4}+\frac{9x}{16}$

$$= \frac{10x+12x+9x}{16}$$

$$= \frac{31x}{16}$$

Total amount of water $= \frac{3x}{8}+\frac{x}{4}+\frac{7x}{16}$

$$= \frac{6x+4x+7x}{16}$$

$$= \frac{17x}{16}$$

∴ Ratio of spirit and water in the big jar $= \frac{31x}{16}:\frac{17x}{16}$

$$= \frac{\frac{31x}{16}}{\frac{17x}{16}}$$

$$= \frac{31}{17} = 31:17$$

Ans: 31 : 17

Example 12 : A grocer sells one quality of tea at Rs. 28 per kg and sustains a loss of 12½% and another quality at Rs. 24 per kg and gains 20%. In what ratio must he mix these two qualities so that the mixture may be sold at Rs. 30 per kg and a profit of 25% is made on the outlay?

Solution: 1st quality of tea

S.P. = Rs. 28 per kg, Loss = 12½%

Loss = 12½% means if C.P. = Rs. 100 then S.P. = Rs. 87½ = Rs. $\frac{175}{2}$

S.P.	C.P
$\frac{175}{2}$	100
28	x

$$x = \text{Rs.} \left(100 \times \frac{28}{\frac{175}{2}} \right)$$

$$= \text{Rs. } \overset{4}{\cancel{100}} \times \overset{4}{\cancel{28}} \times \frac{2}{\underset{7}{\cancel{175}}}$$

= Rs. 32

∴ C.P. of the 1st kind = Rs. 32

2nd quality of tea

S.P. = Rs. 24 per kg, gain = 20%

Gain = 20% means if C.P. = Rs. 100 then S.P. = Rs. 120

S.P.	C.P
120	100
24	y

$$y = \text{Rs.} \left(10\cancel{0} \times \frac{\cancel{24}^{2}}{\cancel{120}} \right)$$

= Rs. 20

∴ C.P. of the 2nd kind = Rs. 20

Now S.P. of the mixture = Rs. 30 per kg and profit = 25%

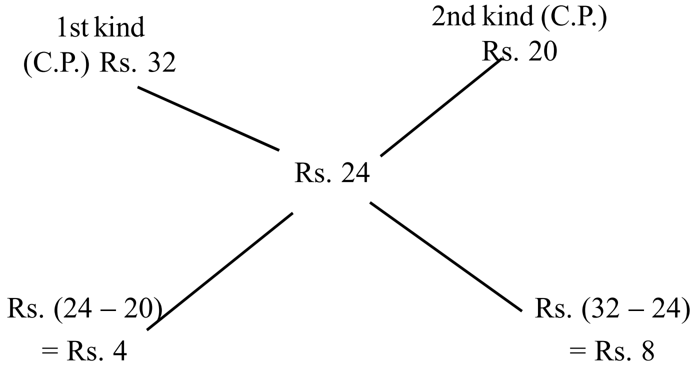
∴ C.P. = Rs. 100, S.P. = Rs. 125

S.P.	C.P
1205	100
30	?

$$? = \text{Rs.} \left(\overset{4}{\cancel{100}} \times \frac{\overset{6}{\cancel{30}}}{\underset{5}{\cancel{125}}} \right)$$

= Rs. 24

∴ C.P. of the mixture = Rs. 24



1st kind : 2nd kind $= 4 : 8$

$$= \frac{4}{8} = \frac{1}{2} = 1:2$$

Required ratio = 1 : 2.

Example 13: A trader mixes 100 kg of rice at one price with 50 kg of rice at a dearer price. By selling the mixture at Rs. 14.52 per kg he makes a profit of 10% on his outlay. Find the price of each kind of rice, the difference in their prices being Rs. 1.20 per kg.

Solution:

S.P. of the mixture per kg = 14.52

Profit = 10%

C.P. = Rs. 100, S.P. = Rs. 110

S.P.	C.P.
110	100
14.52	x

$$x = \text{Rs.} \left(100 \times \frac{14.52}{110}\right)$$

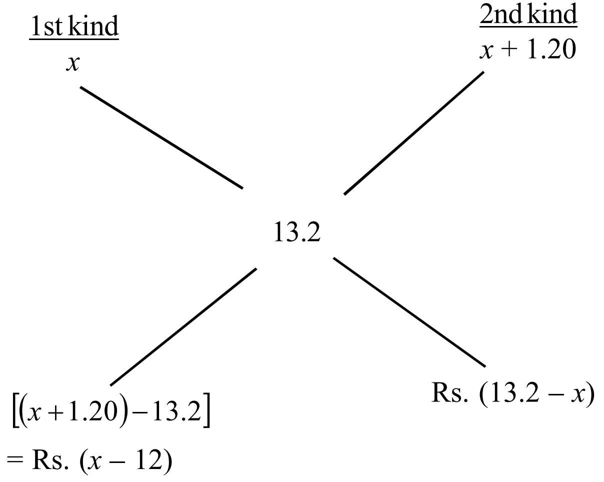
$$= \text{Rs.} \ \frac{\cancel{1452}^{\,132}}{\cancel{110}_{\,10}}$$

$$= \text{Rs. } 13.2$$

$\therefore$ C.P. of the mixture per kg = Rs. 13.2

Let C.P. of the 1st kind = Rs. x per kg

and C.P. of the 2nd kind = Rs. $(x + 1.20)$ per kg



$\therefore$ 1st kind : 2nd kind $= (x - 12) : (13.2 - x)$

$$\Rightarrow \frac{100}{50} = \frac{x-12}{13.2-x}$$

$\Rightarrow \frac{2}{1} = \frac{x-12}{13.2-x}$

$\Rightarrow 26.4 - 2x = x - 12$

$\Rightarrow x + 2x = 26.4 + 12$

$\Rightarrow 3x = 38.4$

$\Rightarrow x = \frac{38.4}{3} = 12.80$

$\therefore$ C.P. of the 1st kind = Rs. 12.80 per kg
C.P. of the 2nd kind = Rs. (12.80 + 1.20)
= Rs. 14.00 per kg

Example 14: Two mixtures A and B contain glycerine and water in the ratio 4 : 5 and 3 : 2 respectively. How many litres of B must be mixed with 54 litres of A so that the resulting mixture may contain equal quantities of glycerine and water?

Solution: In the mixture A
Glycerine : Water = 4 : 5
Amount of mixture = 54 litres

$\therefore$ In the mixture A

Amount of glycerine $= \left(\frac{4}{\cancel{9}} \times \overset{6}{\cancel{54}}\right)$ litres
= 24 litres

and Amount of water $= \left(\frac{5}{\cancel{9}} \times \overset{6}{\cancel{54}}\right)$ litres = 30 litres

In the mixture B
Glycerine : Water = 3 : 2

Let in the mixture B
Amount of glycerine = $3x$
and amount of water = $2x$
$(24 + 3x) = (30 + 2x)$

$\Rightarrow 3x - 2x = 30 - 24$

$\Rightarrow x = 6$ litres

$\therefore$ The mixture B contains (3×6) litre = 18 litres glycerine

and the mixture B contains (2×6) litre = 12 litres water

$\therefore$ Quantity in mixture B
= (18 + 12) litres = 30 litres

$\therefore$ 30 litres of B must be mixed with 54 litres of A.

Exercise

1. Zinc and copper are in the ratio 5 : 3 in 200 gm of an alloy. How much gm of copper be added to make the ratio as 3 : 5?
2. Tea at Rs. 126 per kg and at Rs. 135 per kg are mixed with a third variety in the ratio 1 : 1 : 2. If the mixture is worth Rs. 153 per kg, find the price of the third variety per kg.
3. A shopkeeper sells milk which contains 5% water. What quantity of pure milk should be added to 2 litres of milk (containing 5% water) so that proportions of water becomes 4%?
4. A grocer purchased 20 kg of rice at the rate of Rs. 15 per kg and 30 kg of rice at the rate of Rs. 13 per kg. At what price per kg should he sell the mixture to earn $33\frac{1}{3}\%$ profit on the cost price?
5. A merchant buys sugar at Rs. 4.10, Rs. 3.75 and Rs. 4.50 per kg and mixes in the proportion 5 : 4 : 1. At what price must he sell the mixture so as to gain 25%?
6. In mixing tea, 1 kg in every 100 kg is wasted. In what proportion must a dealer mix tea which cost him Rs. 24 and Rs. 18 per kg respectively, so that the cost is Rs. 20 per kg?
7. Two vessels contain mixture of milk and water in the ratio 5 : 1 and 9 : 1 respectively. In what ratio should the two mixtures be mixed so that a new mixture containing milk and water in the ratio 8 : 1 is formed?
8. A mixture of milk and water contains 12½% water. How much water should be added to 300 litres of such mixture so that the new mixture may contain 37½% water?
9. A milkman mixes 32 litres of water with 168 litres of milk costing Rs. 15 per litre. At what price should he sell per litre mixture so as to make one third profit on his outlay?
10. A trader mixed tea costing Rs. 100 per kg with tea costing Rs. 75 per kg. By selling the mixture at Rs. 108 per kg he made a profit of 20%. In what ratio did he mix the two kind?
11. Dipak mixes 200 kg of flour at one price with 100 kg of flour at a cheaper price. By selling the mixture at Rs. 16.25 per kg he makes a profit of 25% on his outlay. Find the price of each kind if the difference in their prices is Rs. 1.50.
12. In a liquid mixture 20% is water, and in another mixture water is 25%. These two mixtures are mixed in the ratio 5 : 3. Find the percentage of water in the final mixture.

Answers

1. $133\frac{1}{3}$
2. 175.50 per kg
3. 0.5 litres
4. Rs. 18.40
5. Rs. 5.00
6. 3 : 7
7. 1 : 5
8. 120 litres
9. Rs. 16.80
10. 3 : 2
11. Rs. 13.50; Rs. 12.00
12. $21\frac{7}{8}$

Unit II : Sets and Linear Functions

Sets

1.1 Introduction :

The notion of a set is most fundamental in mathematics. The theory of set was originated by the German mathematician Georg Cantor in 1895. In this chapter we will discuss some basic definitions, examples and different operations of sets with some applications.

1.2 Definition :

A set is a well-defined collection of objects. The objects are called elements or members of the set.

Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc. Similarly lower case letters a, b, c, x, y, z, etc. are ordinarily used to denote the elements of a set.

Now we examine the following collections:

(i) The collection of the vowels in English alphabet.

(ii) The collection of the prime numbers less than 20.

(iii) The collection of all girls in your class.

(iv) The collection of all the solutions of the quadratic equation $x^2 - 4x + 4 = 0$

Every collection mentioned above is a well-defined collection of objects because we can decide whether a given particular object belongs to a given collection or not. So every collection is a set.

Next we consider the following collections:

(i) The collection of good cricket players in India.

(ii) The collection of ten most talented writers in India.

(iii) The collection of most dangerous animals of the world.

Since a good cricket player cannot be defined, therefore the collection in (i) is not a set. Similarly the other two collections in (ii) and (iii) are also not sets.

Some standard sets in Mathematics are as follows :

N : the set of natural numbers.

W : the set of whole numbers.

Z : the set of integers.

R : the set of real numbers.

Z^+ : the set of positive integers.

Q^+ : the set of positive rational numbers.

R^+ : the set of positive real numbers.

1.3 Representation of a set :

There are two methods of representing a set :

(i) Roster form or tabular form.

(ii) Set-builder form.

(i) **Roster form or tabular form :** In this form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }. Some examples of representing a set in roster form are as follows :

(a) The set of all vowels in the word 'EQUATION' is {A, E, I, O, U}.

(b) The set of natural numbers less than 7 is {1, 2, 3, 4, 5, 6}.

(c) The set of all prime numbers less than 7 is {2, 3, 5, 7, 11, 13}.

(d) The set of solution of the equation $x^2 = 4$ is {–2, 2}.

(ii) **Set-builder form :** In this form, all the elements of a set possess a single common property which is not possessed by any element outside the set. In other words a set is described by a property $p(x)$ of its elements x. In such a case the set is written as $\{x \mid p(x)$ holds$\}$. Some examples of representing a set in set-builder form are as follows :

(a) Let A = {2, 3, 5}. Then A can be written in set-builder form as

A = $\{x \mid x$ is a prime number less than 7}.

(b) Let B = {4, 5, 6, 7, 8, 9}.

Then B can be written in set-builder form as B = $\{x \mid x$ is a natural number and $3 < x < 10$).

(c) Let C = {2, 4, 6, ...}. Then C can be written in set-builder form as

C = $\{x \mid x$ is an even natural number).

1.4 Notation :

If x is an element of a set X, then we write $x \in X$ and say x belongs to X or x is in X. Thus, if V is the set of all vowels in English alphabet, then $a \in V$ but $d \notin V$. If x is not a member of a set X, then we write $x \notin X$.

1.5 Definition :

A set is said to be empty or null or void set if it has no element. An empty set is generally denoted by φ. In roster form an empty set is denoted by { }. Some examples of empty set as follows :

(i) Let A = $\{x \mid 1 < x < 2$ and x is a natural number).

Then A is an empty set because there is no natural number between 1 and 2.

(ii) Let B = $\{x \mid x$ is an even prime number greater than 2}.

Then B is an empty set because 2 is the only even prime number.

(iii) Let C = the set of all odd natural numbers divisible by 2. We know that no odd natural number is divisible by 2. So C is an empty set.

1.6 Finite and infinite sets :

A set which is empty or has finite number of elements is called finite otherwise, the set is called infinite.

We now consider the following examples :

(i) Let A be the set of days of the week. Then A is a finite set because it has seven distinct members.

(ii) Let B be the set of months of the year. Then B has twelve element and so it is a finite set.

(iii) Let C be the set of solutions of the equation $x^2 - 7x + 12 = 0.$ We know that a quadratic equation has exactly two solutions. So the set C is finite.

(iv) Let D be the set of points on a line. We know that the number of points on a line is infinite and so D is an infinite set.

(v) Let E be the set of all odd prime numbers. There are infinitely many odd prime numbers, viz 3, 5, 7, 11, ... So the set E is an infinite set.

(vi) Let F be the set of all even prime numbers. There is only one even prime number and it is 2. So the set F is a finite set.

1.7 Singleton set : A set is said to be a singleton set if it contains only one element. For example, {6} is a singleton set.

1.8 Equal sets :

Two sets A and B are said to be equal if every element of A is in B and if every element of B is in A, that is they have exactly the same elements. Otherwise, the sets are said to be unequal. If two sets A and B are equal, then symbolically we write A = B. Otherwise, we write $A \neq B$.

Let us consider the following examples:

(i) Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B because A and B have the same elements.

(ii) Let A be the set of even prime number and B be the set of solution of the equation $x–2 = 0$. Then A = {2} and $B = \{x \mid x - 2 = 0\}$

$$= \{x \mid x = 2\}$$

$$= \{2\}$$

So A = B

(iii) Let $X = \{x \mid x$ is a letter in the word 'LOYAL'}

and $Y = \{x \mid x$ is a letter in the word 'ALLOY'}

Then X = {L, O, Y, A) and Y = {L, O, Y, A). So X = Y.

Example 1 : Which of the following are sets and which are not? Justify your answer.

(i) The collection of all boys in your class.

(ii) The collection of difficult topics in mathematics.

(iii) The collection of all natural numbers less than 10.

(iv) The collection of all days of a week beginning with the letter S.

Solution :

(i) The boys in my class are well-defined and so the given collection is a set.

(ii) It is not possible to define which topics in mathematics are difficult and which are easy. So the given collection is not a set.

(iii) The natural numbers less than 10 are exactly known and they are 1, 2, 3, 4, 5, 6, 7, 8, 9. So the given collection is a set.

(iv) The days beginning with the letter S in a week are Sunday and Saturday and so the given collection is a set.

Example 2 : Write the following sets in roster form.

(i) The set of all solutions of the equation $x^2 - 5x + 6 = 0$

(ii) A = $\{x / x$ is a positive integer and $x^2 < 50\}$

(iii) The set of all letters in the word 'MATHEMATICS'.

(iv) The set of all prime numbers less than or equal to 11.

(v) B = $\{x \mid x$ is a two digits natural number such that the sum of its digits is 7$\}$

Solution : (i) Let X be the set of all solutions of the equation $x^2 - 5x + 6 = 0$.

Now, $x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 3x - 3x + 6 = 0$

$\Rightarrow x(x-3) - 2(x-3) = 0$

$\Rightarrow (x-3)(x-2) = 0$

$\Rightarrow x = 3$ or 2

$\therefore$ X = $\{2,3\}$

(ii) Here, A = $\{x \mid x$ is a positive integer and $x^2 < 50\}$

A = {1, 2, 3, 4, 5, 6, 7}

(iii) Let Y be the set of letters in the word 'MATHEMATICS'.

$\therefore$ Y = {M, A, T, H, E, I, C, S}

(iv) Let Z be the set of all prime numbers less than or equal to 11. The prime numbers less than or equal to 11 are 2, 3, 5, 7 and 11.

$\therefore$ Z = {2, 3, 5, 7, 11}

(v) Here,

B = $\{x \mid x$ is a two digits natural number such that the sum of its digits is 7$\}$

$= \{16, 25, 34, 43, 52, 61, 70\}$

Example 3 : Write the following sets in set-builder form.

(i) The set of all letters in the word 'MONDAY'.

(ii) The set of reciprocals of natural numbers.

(iii) $\{2, 4, 8, 16, 32\}$

(iv) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.\right\}$

(v) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots.\right\}$

(vi) $\{0\}$

Solution : (i) Let A be the set of all letters in the word 'MONDAY'. Then

A = $\{x \mid x$ is a letter in the word 'MONDAY'$\}$

(ii) Let B be the set of reciprocals of natural numbers. Then

$$B = \left\{x \mid x = \frac{1}{n}, n \in \mathrm{N}\right\}$$

(iii) Let C = {2, 4, 8, 16, 32)

$$= \{x \mid x = 2^n, n \in \mathrm{N} \text{ and } 1 \leq n \leq 5\}$$

(iv) Let D = $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.\right\}$

$$= \left\{x \mid x = \frac{n}{n+1}, n \in \mathrm{N}\right\}$$

(v) Let E = $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots.\right\}$

$$= \left\{x \mid x = \frac{1}{2n}, n \in \mathrm{N}\right\}$$

(v) Let F = {0}

$$= \{x \mid x = 0\}$$

Example 4 : Find all the elements of the following sets.

(i) $\{x \mid x$ is a positive integer and divisor of 9$\}$

(ii) $\{x \mid x \in \mathrm{N}\}$ and $1.2 < x \leq 8.5\}$

(iii) $\{x \mid x$ is an integer and $x^2 \le 16\}$

(iv) $\left\{x \mid x = \dfrac{n}{n^2+1}\right\}$ and $1 \le n \le 4,$ where $n \in \mathrm{N}\}$

(v) $\{x \mid x$ is a month of a year having 28 or 29 days$\}$

Solution : (i) Let A = $\{x \mid x$ is a positive integer and divisor of 9).

The positive divisors of 9 are 1, 3 and 9.

$\therefore$ A = {1, 3, 9}

(ii) Let B = $\{x \mid x \in \mathrm{N}$ and $1.2 < x \le 8.5\}$.

The natural numbers which are greater than 1.2 and less than or equal to 8.5 are 2, 3, 4, 5, 6, 7 and 8.

$\therefore$ B = {2, 3, 4, 5, 6, 7, 8}

(iii) Let C = $\{x \mid x$ is an integer and $x^2 \le 16)$.

If $x \ge 5,$ then $x^2 \ge 25$ and so $x^2 > 16.$

Again if $x \le -5,$ then also $x^2 \ge 25,$ and so $x^2 > 16.$

And for all other integers $x^2 \le 16.$

$\therefore$ C = $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$

(iv) Let D = $\left\{x \mid x = \dfrac{n}{n^2+1}\right.$ and $1 \le n \le 4,$ where $\left. n \in \mathrm{N}\right\}$

$$= \left\{x \mid x = \frac{1}{1^2+1}, \frac{2}{2^2+1}, \frac{3}{3^2+1} \text{and} \frac{4}{4^2+1}\right\}$$

$$= \left\{x \mid x = \frac{1}{2}, \frac{2}{5}, \frac{3}{10} \text{and} \frac{4}{17}\right\}$$

$$= \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}\right\}$$

(v) Let E = $\{x \mid x$ is a month of a year having 28 or 29 days$\}$

We know that only month having 28 or 29 days is February. So E = {February}.

Example 5 : Which of the following sets are empty sets?

(i) A = $\{x \mid x^2 - 3 = 0$ and x is rational$\}$

(ii) B = $\{x \mid 4 < x < 5$ and $x \in \mathrm{N}\}$

(iii) Set of all even prime numbers.

(iv) C $\{y \mid y$ is a common point to any two different parallel lines$\}$

Solution : (i) Here, A = $\{x \mid x^2 - 3 = 0$ and x is rational$\}$

Now, $x^2 - 3 = 0 \Rightarrow x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$

But we know that $\sqrt{3}$ is an irrational number.

$\therefore$ A $= \phi$

(ii) Here, B = $\{x \mid 4 < x < 5$ and $x \in \mathrm{N}\}$

There is no natural number between 4 and 5 and so B is an empty set.

(iii) Let X = the set of all even prime numbers. We know that 2 is the only even prime number and so X = {2}.

$\therefore$ X is not an empty set.

(iv) Here, C = $\{y \mid y$ is a common point to any two different parallel lines$\}$

There is no common point of two different parallel lines. So C is an empty set.

Example 6 : State which of the following sets are finite and which are infinite.

(i) A = $\{x \mid x \in \mathrm{Z}$ and $x^2 - 9x + 20 = 0\}$

(ii) B = $\{x \in \mathrm{N} \mid x > 5\}$

(iii) The set of days of a week.

(iv) The set of concentric circles in a plane.

Solution : (i) Here,

A = $\{x \mid x \in \mathrm{Z}$ and $x^2 - 9x + 20 = 0\}$

Now, $x^2 - 9x + 20 = 0$

$\Rightarrow x^2 - 5x - 4x + 20 = 0$

$\Rightarrow x(x-5) - 4(x-5) = 0$

$\Rightarrow (x-5)(x-4) = 0$

$\Rightarrow x = 5$ or 4

$\therefore$ A = {5, 4} and so A is a finite set.

(ii) Here, B = $\{x \in \mathrm{N} \mid x > 5\}$

There are infinitely many natural numbers greater than 5.

$\therefore$ B = {6, 7, 8, ...} and it is an infinite set.

(iii) Let C be the set of days of a week. There are seven days in a week and so C has seven elements.

$\therefore$ C is a finite set.

(iv) Let D be the set of concentric circles in a plane. There are infinite number of circles with same centre in a plane. So D is an infinite set.

Example 7 : Are the following pair of sets equal? Justify.

(i) A = {2, 5), B = (x | x is a solution of $x^2 - 7x + 10 = 0\}$

(ii) A = {1, 2), B = $\{x \mid x \in$ N and $x < 3\}$

(iii) A = {x | x is a positive multiple of 10}

B = {10, 15, 20, 25, 30, ...}

Solution : (i) Here, A = (2, 5}

and B = {x | x is a solution of $x^2 - 7x + 10 = 0\}$

Now, $x^2 - 7x + 10 = 0$

$\Rightarrow x^2 - 2x - 5x + 10 = 0$

$\Rightarrow x(x-2) - 5(x-2) = 0$

$\Rightarrow (x-2)(x-5) = 0$

$\Rightarrow x = 2$ or 5

$\therefore$ B = {2, 5}

Here A = B.

(ii) Here, A = {1, 2) and

B = $\{x \mid x \in$ N and $x < 3\}$

= {1, 2}

$\therefore$ A = B

(iii) Here, A = {x | x is a positive multiple of 10}

= {10, 20, 30, 40, ...}

and B = {10, 15, 20, 25, 30, ...}

$\therefore$ A $\neq$ B

Example 8 : Find the pair of equal sets from the following sets and give reasons.

A = {0), B = {x | x > 15 and x < 5}

C = {x | x – 5 = 0}, D = $\{x \mid x^2 = 25\}$

and E = {x | x is a positive integral root of $x^2 - 2x - 15 = 0\}$

Solution : Here, A = (0}

B = {x | x > 15 and x < 5),C = {x | x – 5 = 0}, E = $\{x \mid x^2 = 25\}$ and

E = {x | x is a positive integral root of $x^2 - 2x - 15 = 0\}$

There is no real number which is greater than 15 and less than 5.

$\therefore$ B $= \phi$

We have, $x - 5 = 0 \Rightarrow x = 5$

$\therefore$ C = {5}

Again, $x^2 = 25 \Rightarrow x = \pm 5$

$\therefore$ D = {–5, 5}

Also, $x^2 - 2x - 15 = 0$

$\Rightarrow x^2 - 5x + 3x - 15 = 0$

$\Rightarrow x(x-5) + 3(x-5) = 0$

$\Rightarrow (x-5)(x+3) = 0$

$\Rightarrow x = 5$ or – 3

$\therefore$ E = {5}

Hence C = E

Exercise 1.1

1. Which of the following collections are sets? Justify your answer.
   - (i) The collection of good hockey players in India.
   - (ii) The collection of all districts in Assam.
   - (iii) The collection of beautiful cities in the world.
   - (iv) The collection of all warm days in the year 2023.
   - (v) The collection of all rivers in India.
   - (vi) The collection of all natural numbers less than 100.
   - (vii) The collection of all questions in this chapter.
2. Write the following sets in roster form.
   - (i) The set of real roots of $x^2 - 9 = 0$.
   - (ii) The set of all letters of the word 'COLLEGE'.
   - (iii) The set of all sides of the triangle ABC.
   - (iv) The set of all prime numbers les than 8.
   - (v) A ={x | x is an integer and $-\frac{1}{2} < x < \frac{9}{2}\}$
   - (vi) B ={x | x is an odd natural number}
   - (vii) C ={x | x is a consonant in the English alphabet which precedes k)

(viii) D = $\{x \mid x \in \text{N and } x^2 < 38\}$

3. Write the following sets in set-builder form.
   (i) {A, B, C, ..., X, Y, Z}
   (ii) The set of natural numbers less than 12.
   (iii) The set of all factors of 105 excluding 1.
   (iv) $\{5, 25, 125, 625\}$
   (v) {2, 4, 6, 8, ...}
   (vi) $\{1, 2, 3, 4, 5, 6\}$
   (vii) $\{1, 3, 5, 7, 9\}$
4. Find all the elements of the following sets:
   (i) A = $\{x \mid x \in \text{Z and } x^2 \leq 10\}$
   (ii) B = $\left\{x \mid x = \frac{1}{2n-1}, 1 \leq n \leq 5\right\}$
   (iii) C = $\{x$ | x is a letter of the word 'MISSISIPPI'}
   (iv) D = $\{x \mid x$ is an integral divisor of 18}
   (v) E = $\{x \mid x$ is an integral solution of the equation $2x^2 - 7x + 3 = 0\}$
5. Which of the following sets are empty sets?
   (i) The set of natural numbers less than 1.
   (ii) The set of positive odd integers less than 3.
   (iii) The set of deserts in the Himalayan range.
   (iv) The set of odd natural numbers divisible by 2.
   (v) $\{x \mid x$ is a natural number, $x < 5$ and $x > 8\}$.
   (vi) $\{x \mid x^2 - 2 = 0$ and x is rational$\}$.
   (vii) The set of even natural numbers divisible by 5.
6. State whether the following sets are null set or singleton set.
   (i) A = $\{x \mid x \in \text{R and } x \neq x\}$
   (ii) The set of all prime numbers divisible by 2.
   (iii) $\{x \mid x$ is a month in the yer 2009 having 29 days$\}$.
   (iv) $\{x \mid x$ is a positive root of the equation $x^2 - 49 = 0\}$.
7. State which of the following sets are finite or infinite.
   (i) $\{x \mid x \in \text{N and } (x-1)(x-2) = 0\}$

(ii) $\{x \mid x \in \mathrm{N} \text{ and } 2x-1=0\}$

(iii) The set of all odd natural numbers.

(iv) The set of districts in Assam.

(v) The set of real numbers between 0 and 10.

(vi) The set of lines which are parallel to x-axis.

(vii) The set of numbers which are multiple of 5.

(viii) $\{x \mid x \in \mathrm{N} \text{ and } x < 200\}$

8. Which of the following sets are equal?

$\mathrm{A} = \{x \mid x \in \mathrm{N}, x < 3\}$, B = {1, 2}, C = {3, 1)

$\mathrm{D} = \{x \mid x \in \mathrm{N}, x$ is odd and x <5} and

E = {1, 2, 1,1}

9. Which of the following sets are equal?

(i) A = {x | x is a letter of the word 'FOLLOW'}

B = {x | x is a letter of the word 'WOLF'}

(ii) A = {1, 2, 3}

$\mathrm{B} = \{x \mid x \in \mathrm{R} \mid x^2 - 2x + 2 = 0\}$

1.9 Subset and superset :

Let A and B be any two sets. If each element of A is also an element of B, then A is called a subset of B and B is called a superset of A. In symbol, we write $\mathrm{A} \subseteq \mathrm{B}$ and $\mathrm{B} \supseteq \mathrm{A}$ respectively. If every element of A is also an element of B, but there is at least one element in B, which is not an element of A, then the set A is called a proper subset of B and we write $\mathrm{A} \subset \mathrm{B}$. We consider the following example.

(i) Let X = the set of all students in your school.

Y = the set of all students in your class.

We note that every element of Y is also an element of X and so $\mathrm{Y} \subset \mathrm{X}$.

(ii) We know that every natural number is an integer and so $\mathrm{N} \subset \mathrm{Z}$.

(iii) The set Q of rational numbers is a subset of the set R of real numbers, and we write $\mathrm{Q} \subset \mathrm{R}$.

(iv) Let A = {1, 2}, B = {1, 4,8) and C = {1, 2, 4, 6, 8}.

Then $\mathrm{A} \subset \mathrm{C}$, $\mathrm{B} \subset \mathrm{C}$, but $\mathrm{A} \subseteq \mathrm{B}$.

(v) Let A = the set of all letters in English alphabet.

B = the set of vowels in English alphabet.

Then $\mathrm{B} \subset \mathrm{A}$.

Note :

(a) If two sets A and B are equal, then every element of A is in B and every element of B is also in A. So $\mathrm{A} \subseteq \mathrm{B}$ and $\mathrm{B} \subseteq \mathrm{A}$. Conversely if every element of A is in B and every element of B is in A, then the two sets A and B are equal. Thus $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$, where "$\Leftrightarrow$" is a symbol for two way implications, and is usually read as "if and only if" (briefly written as "iff").

(b) From the definition it is clear that $A \subseteq C$. Since the empty set ϕ has no element,we agree to say that ϕ is a subset of every set.

(c) For a non empty set A, there exist at least two subsets ϕ and A of A. These two subsets of A are called improper subsets.

1.10 Subsets of the set of real numbers:

We consider the set R of real numbers. We have,

N = the set of natural numbers
  = {1, 2, 3, 4, ...}
Z = the set of integers
  = $\{0, \pm 1, \pm 2, \pm 3, ...\}$
Q = the set of rational numbers

$$= \left\{ x \mid x = \frac{a}{b},\ a, b \in \mathrm{Z} \text{ and } b \neq 0 \right\}$$

Let T be the set of all irrational numbers. Then T = $\{x \mid x \in \mathrm{R} \text{ and } x \notin Q\}$. Some of the members of T are $\sqrt{2}, \sqrt{3}, \sqrt{5}$ etc. Some of the relations among these subsets are as follows :

$N \subset Z$, $Z \subset Q$, $Q \subset R$, $T \subset R$, $N \not\subset T$

$\therefore\ N \subset Z \subset Q \subset R$

1.11 Power set :

The collection of all subsets of a set A is called the power set of A and it is denoted by P(A). We consider the following examples.

(i) Let A = {1, 2}. Then $P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

(ii) Let A = {a, b, c}. Then

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

1.12 Universal set :

In any discussion in set theory, there always happens to be a set that contains all sets under consideration i.e. it is a super set of each of the given sets. Such a set is called the universal set and it is denoted by U. Thus, a set that contains all sets in a given particular context is called the universal set.

1.13 Operations on set :

(a) **Union of sets** : The union of two sets A and B, denoted by $A \bigcup B$ (read as 'A union

B'), is the set of all elements which belong to either A or B (or both). In symbol,

$A \bigcup B = \{x \mid x \in A \text{ or } x \in B\}$

(b) **Intersection of sets:** The intersection of two sets A and B, denoted by $A \bigcap B$ (read as 'A intersection B'), is the set of all elements which belong to both A and B. in symbol,

$A \bigcap B = \{x \mid x \in A \text{ and } x \in B\}$

(c) **Difference of sets:** The difference of two sets A and B, denoted by A –B (read as 'A minus B'), is the set of all those elements of A which are not elements of B. In symbol,

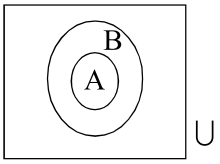
$A - B = \{x \mid x \in A \text{ and } x \in B\}$

(d) **Complement of a set :** The complement of a set A, denoted by A 'or A', is the set of all those elements in the universal set which are not elements of the set A. In symbol

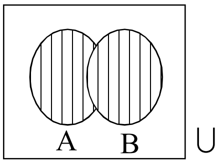
A^c or $A' = \{x \mid x \in U \text{ and } x \notin A\}$

1.13 Venn diagram :

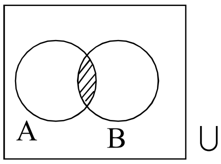
A venn diagram is a representation of sets by means of diagrams. Venn diagrams are named after British mathematician John-Venn (1834-1883). In general, the universal set is represented by the interior of a rectangle and its subsets are represented by circles drawn within the rectangle. Venn diagrams of some set operations are as follows :



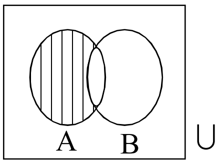
$A \subset B$



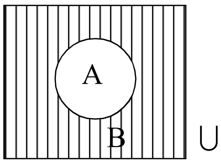
$A \bigcup B$



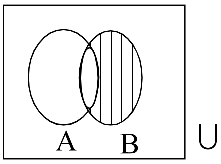
$A \bigcap B$



A – B



A 'or A'



B – A

Example 9 :

(i) If A = {2, 4, 6, 8) and B = {6, 8, 10, 12}, find $A \bigcup B$

(ii) If A = {*a, e, i, o, u*} and B = {*a, i, u*), show that $A \bigcup B = A$

(iii) Let A = {*x* | *x* is a natural number and $| < x \leq 6\}$ and B = {*x* | *x* is a natural number and $6 < x \leq 10\}$ find $A \bigcup B$

(iv) If $A = \{x \mid x = 2n + 1,\ n \in N\}$ and B = $\{x \mid x = 2n,\ n \in N\}$, then find $A \bigcup B$.

Solution: (i) Here, A = {2, 4, 6, 8}
and B = {6, 8, 10,12}

$\therefore\ A \bigcup B = \{2, 4, 6, 8, 10, 12\}$

(ii) Here, A = {*a, e, i, o, u*) and B = {*a, i, u*}

$\therefore\ A \bigcup B = \{a, e, i, o,\ u\}$

$= A$

(iii) Here, A = {*x* | *x* is a natural number and $1 < x \leq 6\}$

= {2, 3, 4, 5, 6}

and B = {x | x is a natural number and $6 < x \leq 10\}$

= {7, 8, 9}

$\therefore\ A \bigcup B = \{2, 3, 4, 5, 6, 7, 8, 9\}$

(iv) Here, $A = \{x \mid x = 2n + 1,\ n \in N\}$

= {3, 5, 7, 9, ...}

and B = $\{x \mid x = 2n,\ n \in N\}$

= {2, 4, 6, 8, ...}

$\therefore\ A \bigcup B = \{2, 3, 4, 5, 6, ...\}$

Example 10 :

(i) If A = {1, 2, 3, 4, 5} and B = {1, 3, 9,12}, find $A \bigcap B$

(ii) If A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and B = (2, 3, 5, 7}, then show that $A \bigcap B = B$

(iii) Let A = $\{x \mid x = 2n,\ n \in Z)$ and B = $\{x \mid x = 3n,\ n \in Z)$, find $A \bigcap B$.

Solution : (i) Here, A = {1, 2, 3, 4, 5} and
B = {1, 3, 9, 12}

$\therefore\ A \bigcap B = \{1, 3\}$

(ii) Here, A= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
and B = {2, 3, 5, 7}

$\therefore\ A \bigcap B = \{2, 3, 5, 7\} = B$

(iii) Here, A = $\{x \mid x = 2n, n \in Z\}$

$= \{0, \pm 2, \pm 4, \pm 6, ...\}$

and B = $\{x \mid x = 3n, n \in Z\}$

$= \{0, \pm 3, \pm 6, \pm 9, \pm 12, ...\}$

$\therefore\ A \bigcap B = \{0, \pm 6, \pm 12, ...\}$

(iv) Here, $A = \{x \mid x \in Z\}$

= {1, 2, 3, 4, 5, ...}

and $B = \{x \mid x = 2n, n \in N\}$

= {2, 4, 6, 8, ...}

$\therefore\ A \bigcap B = \{2, 4, 6, 8, ...\} = B.$

Example 11 :

(i) Let A = {1, 2, 3, 4, 5, 6} and B = {2, 4, 6, 8). Find A – B and B – A

(ii) If X = {*a, b, c, d*} and Y = {*f, b, d, g*}, find X – Y and Y – X.

(iii) If U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {2, 4, 6, 8} and B = {2, 3, 5, 7), find A' and B'.

(iv) If U = {*a, b, c, d, e, f, g, h*}, A = {*a, c, e, g*}, find A^c.

Solution : (i) Here, A = {1, 2, 3, 4, 5, 6} and

B = {2, 4, 6, 8}

∴ A – B = {1, 3, 5} and

B – A = {8}

(ii) Here, X = {a, b, c, d} and

Y = {f, b, d, g}

∴ X – Y = {*a, c*} and Y – X = {*f, g*}

(iii) Here, U = {1, 2, 3, 4, 5, 6, 7, 8, 9}

A = {2, 4, 6, 8}

and B = {2, 3, 5, 7}

$\therefore\ A' = \{1, 3, 5, 7, 9\}$

and $B' = \{1, 4, 6, 8, 9\}$

(iv) Here, U = {*a, b, c, d, e, f, g, h*} and

A = {*a, c, e, g*}

∴ A^c = {*b, d, f, h*}

Example 12: Let U = {1, 2, 3, 4, 5, 6}, A = {1, 2, 3} and B = {3, 4, 5}. Find $A \bigcup B, A \bigcap B$, A – B, B – A and A^c.

Solution : Here U = {1, 2, 3, 4, 5, 6}, A = {1, 2, 3} and B = {3, 4, 5}.

$\therefore \ A \bigcup B = \{1,2,3,4,5\}$

$A \bigcap B = \{3\}$

A – B = {1, 2}

B – A = {4, 5}

and A^c = {4, 5, 6}

1.14 Disjoint sets :

Two sets A and B are said to be disjoint if $A \bigcap B = \phi$. For example, let A = {2, 4, 6, 8} and B = {1, 3, 5, 7}. Then $A \bigcap B = \phi$ and so A and B are disjoint.

Example 13 : Whic of the following pairs of sets are disjoint?

(i) {1, 2, 3, 4} and {x | x is a natural number and $4 \le x \le 6$}

(ii) {*a, e, i, o, u*} and {*c, d, e, f*}

(iii) {x | x is an even positive integer} and {x | x is an odd positive integer}

Solution : (i) Let A = {1, 2, 3, 4} and

B = {x | x is a natural number and $4 \le x \le 6$} = {4, 5, 6}

$\therefore \ A \bigcap B = \{4\} \neq \phi$ and so A and B are two non disjoint sets.

(ii) Let A = {*a, e, i, o, u*}
and B = {*c, d, e, f*}

$\therefore \ A \bigcap B = \{e\} \neq \phi$ and so A and B are not disjoint.

(iii) Let A = {x | x is an even positive integer}
= {2, 4, 6, 8, ...}
and B = {x | x is an odd positive integer}
= {1, 3, 5, 7, ...}

$\therefore \ A \bigcap B = \phi$ and so A and B are two disjoint sets.

Some properties of union:

Theorem 1 : For any three sets A, B and C, prove the following laws:

(i) Commutative law: $A \bigcup B = B \bigcup A$

(ii) Associative law : $A \bigcup (B \bigcup C) = (A \bigcup B) \bigcup C$

(iii) Laws of $\bigcup$ and ϕ : $A \bigcup \phi = A, \ \bigcup \bigcup A = \bigcup$

(iv) Idempotent law : $A \bigcup A = A$

Proof : (i) Let $x \in A \bigcup B$

$\Leftrightarrow x \in A$ or $x \in B$

$\Leftrightarrow x \in$ B or $x \in \mathrm{A}$

$\Leftrightarrow x \in \mathrm{B} \bigcup \mathrm{A}$

$\therefore$ $\mathrm{A} \bigcup \mathrm{B} \subseteq \mathrm{B} \bigcup \mathrm{A}$ and $\mathrm{B} \bigcup \mathrm{A} \subseteq \mathrm{A} \bigcup \mathrm{B}$

Hence, $\mathrm{A} \bigcup \mathrm{B} = \mathrm{B} \bigcup \mathrm{A}$

(ii) Let $x \in \mathrm{A} \bigcup (\mathrm{B} \bigcup \mathrm{C})$

$\Leftrightarrow x \in \mathrm{A}$ or $x \in \mathrm{B} \bigcup \mathrm{C}$

$\Leftrightarrow x \in \mathrm{A}$ or $(x \in \mathrm{B}$ or $x \in \mathrm{C})$

$\Leftrightarrow (x \in \mathrm{A}$ or $x \in \mathrm{B})$ or $x \in \mathrm{C}$

$\Leftrightarrow x \in \mathrm{A} \bigcup \mathrm{B}$ or $x \in \mathrm{C}$

$\Leftrightarrow x \in (\mathrm{A} \bigcup \mathrm{B}) \bigcup \mathrm{C}$

$\therefore$ $\mathrm{A} \bigcup (\mathrm{B} \bigcup \mathrm{C}) \subseteq (\mathrm{A} \bigcup \mathrm{B}) \bigcup \mathrm{C}$ and $(\mathrm{A} \bigcup \mathrm{B}) \bigcup \mathrm{C} \subseteq \mathrm{A} \bigcup (\mathrm{B} \bigcup \mathrm{C})$

Hence, $\mathrm{A} \bigcup (\mathrm{B} \bigcup \mathrm{C}) = (\mathrm{A} \bigcup \mathrm{B}) \bigcup \mathrm{C}$

(iii) Let $x \in \mathrm{A} \bigcup \phi$

$\Leftrightarrow x \in \mathrm{A}$ or $\Leftrightarrow x \in \phi$

$\Leftrightarrow x \in \mathrm{A}$ $[\because x \notin \phi]$

$\therefore$ $\mathrm{A} \bigcup \phi \subseteq \mathrm{A}$ and $\mathrm{A} \subseteq \mathrm{A} \bigcup \phi$

Hence, $\mathrm{A} \bigcup \phi = \mathrm{A}$

Next, let $x \in \mathrm{U} \bigcup \mathrm{A}$

$\Leftrightarrow x \in \bigcup$ or $x \in \mathrm{A}$

$\Leftrightarrow x \in \bigcup$

$\therefore$ $\mathrm{U} \bigcup \mathrm{A} \subseteq \bigcup$ and $\bigcup \subseteq \mathrm{U} \bigcup \mathrm{A}$

Hence, $\mathrm{U} \bigcup \mathrm{A} = \mathrm{U}$

(iv) Let $x \in \mathrm{A} \bigcup \mathrm{A}$

$\Leftrightarrow x \in \mathrm{A}$ or $x \in \mathrm{A}$

$\Leftrightarrow x \in \mathrm{A}$

$\therefore$ $\mathrm{A} \bigcup \mathrm{A} \subseteq \mathrm{A}$ and $\mathrm{A} \subseteq \mathrm{A} \bigcup \mathrm{A}$

Hence, $\mathrm{A} \bigcup \mathrm{A} = \mathrm{A}$

Some properties of intersection:

Theorem 2 : For any three sets, A, B and C, prove the following laws :

(i) Commutative law : $A \bigcap B = B \bigcap A$

(ii) Associative law : $A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C$

(iii) Laws of ϕ and $\bigcup$: $\phi \bigcap A = \phi$, $U \bigcap A = A$

(iv) Indempotent law : $A \bigcap A = A$

(v) Distributive laws: $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$

$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$

Proof: (i) Let $x \in A \bigcap B$

$\Leftrightarrow x \in A$ and $\Leftrightarrow x \in B$

$\Leftrightarrow x \in B$ and $\Leftrightarrow x \in A$

$\Leftrightarrow x \in B \bigcap A$

$\therefore$ $A \bigcap B \subseteq B \bigcap A$ and $B \bigcap A \subseteq A \bigcap B$

Hence, $A \bigcap B = B \bigcap A$

(ii) Let $x \in A \bigcap (B \bigcap C)$

$\Leftrightarrow x \in A$ and $x \in B \bigcap C$

$\Leftrightarrow x \in A$ and $(x \in B \text{ and } x \in C)$

$\Leftrightarrow (x \in A \text{ and } x \in B)$ and $x \in C$

$\Leftrightarrow x \in A \bigcap B$ and $x \in C$

$\Leftrightarrow x \in (A \bigcap B) \bigcap C$

$\therefore$ $A \bigcap (B \bigcap C) \subseteq (A \bigcap B) \bigcap C$ and $(A \bigcap B) \bigcap C \subseteq A \bigcap (B \bigcap C)$

Hence, $A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C$

(iii) There is no element in ϕ and so there is no element common to both A and ϕ.

Hence $\phi \bigcap A = \phi$

Again all the elements of A are in $\bigcup$. So the elements of A are common to both A and $\bigcup$.

Hence $U \bigcap A = A$

(iv) Let $x \in A \bigcap A$

$\Leftrightarrow x \in A$ and $x \in A$

$\Leftrightarrow x \in A$

$\therefore\ A \cap A \subseteq A$ and $A \subseteq A \cap A$

Hence, $A \cap A = A$

(v) Let $x \in A \cap (B \cap C)$

$\Leftrightarrow x \in A$ and $\Leftrightarrow x \in B \cup C$

$\Leftrightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Leftrightarrow (x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \in C)$

[$\because$ 'and' is distributive over 'or']

$\Leftrightarrow x \in A \cap B$ or $x \in A \cap C$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

$\therefore\ A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Similarly, we can prove that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Some properties of complement:

Theorem 3: For any two sets A and B, prove the following laws:

(i) Complement laws : $A \cup A' = U,\ A \cap A' = \phi$

(ii) De Morgan's laws : $(A \cup B)' = A' \cap B',\ (A \cap B)' = A' \cup B'$

(iii) Law of double complementation : $(A')' = A$

(iv) Laws of ϕ and $\bigcup$: $\phi' = \bigcup,\ \bigcup' = \phi$

Proof : (i) We have

$A \cup A' = \{x \in \bigcup \mid x \in A\} \cup \{x \in \bigcup \mid x \in A'\}$

$= \{x \in \bigcup \mid x \in A\} \cup \{x \in \bigcup \mid x \notin A\}$

$= \bigcup$

and $A \cap A' = \{x \in \bigcup \mid x \in A\} \cap \{x \in \bigcup \mid x \notin A'\}$

$= \{x \in \bigcup \mid x \in A\} \cap \{x \in \bigcup \mid x \notin A\}$

$= \phi$

(ii) Let $x \in (A \cup B)'$

$\Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A$ and $\Leftrightarrow x \notin B$

$\Leftrightarrow x \in A'$ and $x \in B'$

$\Leftrightarrow x \in A' \bigcap B'$

$\therefore$ $(A \bigcup B)' \subseteq A' \bigcap B'$ and $A' \bigcap B' \not\subset (A \bigcup B)'$

Hence, $(A \bigcup B)' = A' \bigcap B'$

Next, let $x \in (A \bigcap B)'$

$\Rightarrow x \notin A \bigcap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \bigcup B'$

$\therefore (A \bigcap B)' \subseteq A' \bigcup B'$ and $A' \bigcup B' \subseteq (A \bigcap B)'$

Hence, $(A \bigcap B)' = A' \bigcup B'$

(iii) Let $x \in (A')'$

$\Leftrightarrow x \notin A'$

$\Leftrightarrow x \in A$

$\therefore (A')' \subseteq A$ and $A \subseteq (A')'$

Hence, $(A')' = A$

(iv) We have

$\phi' = \{x \in \bigcup \mid x \notin \phi\} = \bigcup$

and $\bigcup' = \{x \in \bigcup \mid x \notin \bigcup\} = \phi$

Example 14 : If $\bigcup = \{1,2,3,4,5,6,7,8,9\}$, A = {2, 4, 6, 8} and B = {2, 3, 5, 7}, verity that $(A \bigcup B)' = A' \bigcap B'$ and $(A \bigcap B)' = A' \bigcup B'$

Solution: Here, $\bigcup = \{1,2,3,4,5,6,7,8,9\}$,

A = {2, 4, 6, 8}

and B = {2, 3, 5, 7}

$\therefore$ $A' = \{1,3,5,7,9\}$

$B' = \{1,4,6,8,9\}$

$A' \bigcap B' = \{1,9)$

$A' \bigcup B' = \{1,3,4,5,6,7,8,9\}$

Again, $A \bigcap B = \{2\}$

$A \bigcup B = \{2,3,4,5,6,7,8\}$

$\therefore \quad (A \bigcap B)' = \{1,3,4,5,6,7,8,9)$

$(A \bigcup B)' = \{1, 9)$

Here $(A \bigcup B)' = A' \bigcap B'$ and $(A \bigcap B)' = A' \bigcup B'$

Example 15 : For any three sets A, B, and C, prove that

(i) $A - B = A \bigcap B^c$

(ii) $(A - B) \bigcap B = A \bigcup B$

(iii) $A - (B \bigcup C) = (A - B) \bigcap (A - C)$

(iv) $A - (B \bigcap C) = (A - B) \bigcup (A - C)$

Solution: (i) Let $x \in A - B$

$\Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow x \in A$ and $x \in B^c$

$\Leftrightarrow x \in A \bigcap B^c$

$\therefore \quad A - B \subseteq A \bigcap B^c$ and $A \bigcap B^c \subseteq A - B$

Hence, $A - B = A \bigcap B^c$

(ii) Let $x \in (A - B) \bigcup B$

$\Leftrightarrow x \in A - B$ or $x \in B$

$\Leftrightarrow (x \in A$ and $x \notin B)$ or $x \in B$

$\Leftrightarrow (x \in A$ and $x \in B^c)$ or $x \in B$

$\Leftrightarrow (x \in A \text{ or } x \in B)$ and $(x \in B^c \text{ or } \in B)$

$\Leftrightarrow x \in A \bigcup B$ and $x \in B^c \bigcup B$

$\Leftrightarrow x \in A \bigcup B$ and $x \in \bigcup$

$\Leftrightarrow x \in (A \bigcup B) \bigcap U$

$\Leftrightarrow x \in A \bigcup B$

$\therefore \quad (A - B) \bigcup B \subseteq A \bigcup B$ and $A \bigcup B \subseteq (A - B) \bigcup B$

Hence, $(A - B) \bigcup B = A \bigcup B$

(iii) Let $x \in A - (B \bigcup C)$

$\Leftrightarrow x \in A$ and $x \notin B \bigcup C$

$\Leftrightarrow x \in \mathrm{A}$ and $(x \notin \mathrm{B}$ and $x \notin \mathrm{C})$

$\Leftrightarrow (x \in \mathrm{A}$ and $x \notin \mathrm{B})$ and $(x \in \mathrm{A}$ and $x \notin \mathrm{C})$

$\Leftrightarrow x \in \mathrm{A} - \mathrm{B}$ and $x \in \mathrm{A} - \mathrm{C}$

$\Leftrightarrow x \in (\mathrm{A} - \mathrm{B}) \bigcap (\mathrm{A} - \mathrm{C})$

$\therefore$ $\mathrm{A} - (\mathrm{B} \bigcup \mathrm{C}) \subseteq (\mathrm{A} - \mathrm{B}) \bigcap (\mathrm{A} - \mathrm{C})$ and

$(\mathrm{A} - \mathrm{B}) \bigcap (\mathrm{A} - \mathrm{C}) \subseteq \mathrm{A} - (\mathrm{B} \bigcup \mathrm{C})$

Hence, $\mathrm{A} - (\mathrm{B} \bigcup \mathrm{C}) = (\mathrm{A} - \mathrm{B}) \bigcap (\mathrm{A} - \mathrm{C})$

(iv) Let $x \in \mathrm{A} - (\mathrm{B} \bigcap \mathrm{C})$

$\Leftrightarrow x \in \mathrm{A}$ and $x \in \mathrm{B} \bigcap \mathrm{C}$

$\Leftrightarrow x \in \mathrm{A}$ and $(x \in \mathrm{B}$ or $x \notin \mathrm{C}$

$\Leftrightarrow (x \in \mathrm{A}$ and $x \notin \mathrm{B})$ or $(x \in \mathrm{A}$ and $x \notin \mathrm{C})$

$\Leftrightarrow x \in \mathrm{A} - \mathrm{B}$ or $x \in \mathrm{A} - \mathrm{C}$

$\Leftrightarrow x \in (\mathrm{A} - \mathrm{B}) \bigcup (\mathrm{A} - \mathrm{C})$

$\therefore$ $\mathrm{A} - (\mathrm{B} \bigcap \mathrm{C}) \subseteq (\mathrm{A} - \mathrm{B}) \bigcup (\mathrm{A} - \mathrm{C})$

and $(\mathrm{A} - \mathrm{B}) \bigcup (\mathrm{A} - \mathrm{C}) \subseteq \mathrm{A} - (\mathrm{B} \bigcap \mathrm{C})$

Here, $\mathrm{A} - (\mathrm{B} \bigcap \mathrm{C}) = (\mathrm{A} - \mathrm{B}) \bigcup (\mathrm{A} - \mathrm{C})$

Example 16 : Let $\bigcup = \{1, 2, 3, 4, 5, 6\}, \mathrm{A} = \{1, 2, 3\}, \mathrm{B} = \{2, 3, 4\}$ and $\mathrm{C} = \{3, 4, 5\}$. Verify the following identities:

(i) $\mathrm{A} \bigcup (\mathrm{B} \bigcap \mathrm{C}) = (\mathrm{A} \bigcup \mathrm{B}) \bigcap (\mathrm{A} \bigcup \mathrm{C})$

(ii) $\mathrm{A} \bigcap (\mathrm{B} \bigcup \mathrm{C}) = (\mathrm{A} \bigcap \mathrm{B}) \bigcap (\mathrm{A} \bigcap \mathrm{C})$

Solution: Here, $\bigcup = \{1, 2, 3, 4, 5, 6\}, \mathrm{A} = \{1, 2, 3\}, \mathrm{B} = \{2, 3, 4\}$ and $\mathrm{C} = \{3, 4, 5\}$

(i) We have

$\mathrm{B} \bigcap \mathrm{C} = \{3, 4\}$

$\mathrm{A} \bigcup (\mathrm{B} \bigcap \mathrm{C}) = \{1, 2, 3, 4\}$

Again, $\mathrm{A} \bigcup \mathrm{B} = \{1, 2, 3, 4\}$

$\mathrm{A} \bigcup \mathrm{C} = \{1, 2, 3, 4, 5\}$

$(\mathrm{A} \bigcup \mathrm{B}) \bigcap (\mathrm{A} \bigcup \mathrm{C}) = \{1, 2, 3, 4\}$

$\therefore$ $\mathrm{A} \bigcup (\mathrm{B} \bigcap \mathrm{C}) = (\mathrm{A} \bigcup \mathrm{B}) \bigcap (\mathrm{A} \bigcup \mathrm{C})$

(ii) We have

$$B \cup C = \{2,3,4,5\}$$

$$A \cap (B \cup C) = \{2,3\}$$

Again $A \cap B = \{2,3\}$

$$A \cap C = \{3\}$$

$$(A \cap B) \cup (A \cap C) = \{2,3\}$$

$$\therefore \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example 17: If $\cup = \{x \mid x$ is a natural number $<6\}$,

A = $\{x \mid x$ is a prime number $<6\}$ and

B = $\{x \mid x$ is an even natural number $<5\}$, find $B' - A'$.

(ii) If A = $\{x \mid x = 2n, n \in N, n < 6\}$, find $A \cup N$ and $A - N$.

Solution: (i) Here,

$\cup = \{x \mid x$ is a natural number $<6\}$

$= \{1, 2, 3, 4, 5\}$

A = $\{x \mid x$ is a prime number $<6\}$

$= \{2, 3, 4\}$

and B = $\{x \mid x$ is an even natural number $<5\}$

$= \{2, 4\}$

$\therefore \quad A' = \{1,4\}$

$B' = \{1,3,5\}$

$\therefore \quad B' - A' = \{3,5\}$

(ii) Here, A = $\{x \mid x = 2n, n \in N, n < 6\}$

$= \{2, 4, 6, 8, 10\}$

$\therefore \quad A \cup N = \{1, 2, 3, 4, 5, ...\} = N$

and $A - N = \phi$

1.15 Symmetric difference of two sets :

The symmetric difference of any two sets A and B, denoted by $A \Delta B$ is defined as

$$A \Delta B = (A - B) \cup (B - A)$$

Example 18 : Let A = $\{a, b, c, d, e\}$, B = $\{a, c, e, g\}$ and C = $\{b, e, f, g\}$. Verify that

(i) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(ii) $A - (B \bigcup C) = (A - B) \bigcap (A - C)$

(iii) $A \bigcap (B \Delta C) = (A \bigcap B) \Delta (A \bigcap C)$

Solution : Here, A = $\{a, b, c, d, e\}$, B = $\{a, c, e, g\}$ and C = $\{b, e, f, g\}$

(i) We have,

$$A \bigcup B = \{a, b, c, d, e, g\}$$

$$(A \bigcup B) \bigcap C = \{b, e, g\}$$

Again $A \bigcap C = \{b, e\}$

$$B \bigcap C = \{e, g\}$$

$$(A \bigcap C) \bigcup (B \bigcap C) = \{b, e, g\}$$

$$\therefore \quad (A \bigcup B) \bigcap C = (A \bigcap C) \bigcup (B \bigcap C)$$

(ii) We have,

$$B \bigcup C = \{a, b, c, e, f, g\}$$

$$A - (B \bigcup C) = \{d\}$$

Again $A - B = \{b, d\}$

$$A - C = \{a, c, d\}$$

$$(A - B) \bigcap (A - C) = \{d\}$$

$$\therefore \quad A - (B \bigcup C) = (A - B) \bigcap (A - C)$$

(iii) We have,

$$B \Delta C = (B - C) \bigcup (C - B)$$

$$= \{a, c\} \bigcup \{b, f\}$$

$$= \{a, b, c, f\}$$

$$A \bigcap (B \Delta C) = \{a, b, c\}$$

Agin $A \bigcap B = \{a, c, e\}$

$$A \bigcap C = \{b, e\}$$

$$(A \bigcap B) \Delta (A \bigcap C) = \big((A \bigcap B) - (A \bigcap C) \bigcup)\big) \bigcup \big((A \bigcap C) - (A \bigcap B)\big)$$

$$= \{a, c\} \bigcup \{b\}$$

$$= \{a, b, c\}$$

$$\therefore \quad A \bigcap (B \Delta C) = (A \bigcap B) \Delta (A \bigcap C)$$

Example 19 : For any two sets A and B, prove that $A \bigcup B = A \bigcap B \Leftrightarrow A = B$

Proof : First suppose that A = B.

$\therefore \quad A \bigcup B = A \bigcup A = A$

and $\quad A \bigcap B = A \bigcap A = A$

$\therefore \quad A \bigcup B = A \bigcap B$

Conversely, suppose that $A \bigcup B = A \bigcap B$

Let, $\quad x \in A$

$\Rightarrow x \in A \bigcup B$

$\Rightarrow x \in A \bigcap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$

$\therefore \quad A \subseteq B$

Next, let $x \in B \Rightarrow x \in A \bigcup B$

$\Rightarrow x \in A \bigcap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A$

$\therefore \quad B \subseteq A$

Hence A = B.

Example 20 : Distinguish between $\{0\}$, ϕ and $\{\phi\}$.

Solution : $\{0\}$ is a singleton set containing the single element 0. ϕ is the null set containing no element. And $\{\phi\}$ is a set of set whose only element is the null set ϕ .

Example 21 : Find the symmetric difference of A = {1, 2, 3} and B = {3, 4, 5}.

Solution : Here, A = {1, 2, 3}

and B = {3, 4, 5}

$\therefore \quad A - B = \{1, 2\}$

$B - A = \{4, 5\}$

$\therefore \quad A \Delta B = (A - B) \bigcup (B - A)$

$= \{1, 2, 4, 5\}$

Example 22 : Let A = $\{x \mid x$ is an integer and $1 \leq x \leq 9\}$

B = $\{x \mid x$ is a prime number and $1 < x < 12\}$

and C = $\{x \mid x$ is an odd integer and $4 < x < 12\}$

Find (i) $A \bigcap (B \bigcap C)$ (ii) $B \Delta C$ (iii) $A - (B \bigcup C)$

(iv) $(A - B) \bigcup (A - C)$ (v) $(A \bigcap B) \bigcup C$

Solution : Here,

$A = \{x \mid x \text{ is an integer and } 1 \le x \le 9\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$B = \{x \mid x \text{ is a prime number and } 1 < x < 12\}$

$= \{2, 3, 5, 7, 11\}$

and $C = \{x \mid x \text{ is an odd integer and } 4 < x < 12\}$

$= \{5, 7, 9, 11\}$

(i) We have

$B \bigcap C = \{5,7,11\}$

$\therefore A \bigcap (B \bigcap C) = \{5,7\}$

(ii) We have,

$B - C = \{2, 3\}$

$C - B = \{9\}$

$\therefore B \Delta C = (B - C) \bigcup (C - B)$

$= \{2,3\} \bigcup \{9\}$

$= \{2, 3, 9\}$

(iii) We have,

$BUC = \{2,3,5,7,9,11\}$

$\therefore A - (B \bigcup C) = \{1,4,6,8\}$

(iv) We have,

$A - B = \{1, 4, 6, 8, 9\}$

$A - C = \{1, 2, 3, 4, 6, 8\}$

$\therefore (A - B) \bigcup (A - C) = \{1, 2, 3, 4, 6, 8, 9\}$

(v) We have,

$A \bigcap B = \{2, 3, 5, 7\}$

$\therefore (A \bigcap B) \bigcup C = \{2, 3, 5, 7, 9, 11\}$

Example 23 : Let $A = \{a, b, \{c, d\}, e\}$. State whether the following statements are true or false. Give reasons.

(i) $\{c, d\} \subseteq A$ (ii) $\{c, d\} \in A$ (iii) $\{\{c,d\}\} \subseteq A$

(iv) $a \in A$ (v) $a \subseteq A$ (vi) $\{a,b,e\} \subseteq A$

(vii) $\{a,b,e\} \in A$ (viii) $\{a,b,c\} \subseteq A$ (ix) $\phi \in A$

(x) $\{\phi\} \subseteq$ A

Solution : Here A = $\{a, b, \{c, d\}, e\}$

(i) $\{c, d\} \subseteq$ A is false because $\{c, d\}$ is an element of A.

(ii) $\{c, d\} \in$ A is true because $\{c, d\}$ is an element of A.

(iii) $\{\{c, d\}\} \subseteq$ A is true because {c, d} is an element of A and so $\{\{c, d\}\}$ is a subset of A.

(iv) $a \in$ A is true because a is an element of A.

(v) $a \subseteq$ A is false because a is not a set.

(vi) $\{a, b, e\} \subseteq$ A is true.

(vii) $\{a, b, e\} \in$ A is false because $\{a, b, e)$ is a set.

(viii) $\{a, b, c\} \subseteq$ A is false because c is not an element of A.

(ix) $\phi \in$ A is false because ϕ is always a subset of any set.

(x) $\{\phi\} \subseteq$ A is false because $\{\phi\}$ is a set of a set and it cannot be a subset of A.

Exercise 1.2

1. State whether A $\subseteq$ B or A $\not\subseteq$ B in each of the following examples.
   (i) A = ϕ, B = {1, 2}
   (ii) A = $\{a, b, c\}$, B = $\{a, b, d, e\}$
   (iii) A = the set of all squares in a plane.
   B = the set of all rectangles in the plane.
   (iv) A = the set of all equilateral triangles in a plane.
   B = the set of all triangles in the plane.
   (v) A = {2, 3, 4}, B = {3, 4, 5, 6}
   (vi) A = the set of all integers.
   B = the set of all even integers.
   (vii) A = $\{x \mid x$ is a circle in a plane with radius 1 unit}
   B = $\{x \mid x$ is a circle in the same plane}
2. If A = {1, 2, 3, 4}, find p(A)
3. Write all the subsets of the set $\{a\}$.
4. Decide, among the following sets, which sets are subsets of one and another.

   A = $\{x \mid x \in$R and $x^2 - 8x + 12 = 0\}$

   B ={2, 4, 6}, C = {2, 4, 6, 8, ...}

and D = {6}.

5. (i) Find A $\bigcup$ B if A = {1, 2, 3} and B = {2, 3, 5, 6}.

(ii) Find $A \bigcup B$ if $A = \{x \mid x$ is a natural number and multiple of $3\}$ and $B = \{x \mid x$ is a natural number less than $6\}$.

6. If $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, $C = \{7, 8, 9, 10, 11\}$ and $D = \{10, 11, 12, 13, 14\}$, find

(i) $A \bigcup B$ (ii) $A \bigcup C$ (iii) $B \bigcup C$ (iv) $B \bigcup D$

(v) $A \bigcup (B \bigcup C)$ (vi) $A \bigcup B \bigcup D$ (vii) $B \bigcup C \bigcup D$ (viii) $A \bigcap (BC)$

(ix) $(A \bigcap B) \bigcap (B \bigcap C)$ (x) $(A \bigcup D) \bigcap (B \bigcup C)$

7. If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, find

(i) $A \bigcap B$ (ii) $B \bigcap C$ (iii) $A \bigcap C \bigcap D$ (iv) $A \bigcap C$ (v) $B \bigcap D$ (vi) $A \bigcap (B \bigcup C)$ (vii) $A \bigcap D$

(viii) $A \bigcap (B \bigcup D)$ (ix) $(A \bigcap B) \bigcap (B \bigcup C)$ (x) $(A \bigcup D) \bigcap (B \bigcup C)$

8. If $A = \{3, 6, 9, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$ and $D = \{5, 10, 15, 20\}$, find (i) $A - B$ (ii) $A - C$ (iii) $A - D$ (iv) $B - A$ (v) $C - A$ (vi) $D - A$ (vii) $B - C$ (viii) $B - D$ (ix) $C - B$ (x) $D - B$ (xi) $C - D$ (xii) $D - C$

9. Let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 5, 6\}$ and $C = \{4, 5, 6, 7\}$ verify the following identities:

(i) $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$

(ii) $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$

(iii) $A \bigcap (B - C) = (A \bigcap B) - (A \bigcap C)$

(iv) $A - (B \bigcup C) = (A - B) \bigcap (A - C)$

(v) $A - (B \bigcap C) = (A - B) \bigcup (A - C)$

(vi) $A \bigcap (B \Delta C) = (A \bigcap B) \Delta (A \bigcap C)$

10. Let $U = \{2, 3, 5, 7, 11, 13\}$, $A = \{5, 7, 13\}$ and $B = \{3, 7, 11, 13\}$

(i) Find $(A - B)'$

(ii) Prove that $(A \bigcup B)' = A' \bigcap B'$

11. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 4, 7, 8\}$ and $B = \{4, 6, 8, 9\}$ and $C = \{3, 4, 5, 7\}$ find

(i) $A \bigcap (B - C)$

(ii) $A' \bigcap (B' - C')$

(iii) $A \Delta C$

(iv) $A \bigcup (B \bigcap C')$

(v) $A \bigcap (B \Delta C)$

12. If $\bigcup = \{2, 3, 5, 7, 9\}$, $A = \{3, 7\}$ and $B = \{2, 5, 7, 9\}$, then prove that (i) $(A \bigcup B)' = A' \bigcap B'$

and (ii) $(A \cap B)' = A' \cup B'$

13. (i) If A = $\{x \mid x \in R \text{ and } 3 < x < 8\}$ and

    B = $\{x \mid x \in R \text{ and } 5 < x < 9\}$, find $A \cap B$.

    (ii) If A = $\{x \mid x^2 - 4 = 0\}$ and B = $\{x \mid x^2 + 4x - 12 = 0\}$, find $A \cap B$.

    (iii) If A = $\{x \mid x \in R \text{ and } 1 \le x \le 7\}$ and B = $\{x \mid x \in R \text{ and } 3 \le x \le 10\}$, find $A \cup B$ and $A \cap B$.

    (iv) If A = $\{x \mid x \in N \text{ and } x^2 + 4x - 12 = 0\}$ and B = $\{x \mid x \in N \text{ and } x^2 - 6x + 8 = 0\}$, find $A \cap B$.

14. Let A = {*a, b*} and B = {*a, b, c*}. Is $A \subset B$? What is $A \cup B$?

15. State whether each of the following statements is true or false. Justify your answer.

    (i) {2, 3, 4} and {3, 4, 5} are disjoint sets.

    (ii) {2, 6, 10} and (3, 7, 11) are disjoint sets.

    (iii) The set of all triangles and the set of all equilateral triangles are disjoint.

16. (i) If A = {2, 3, 4, 5}, B = {3, 6, 9} and C = {5, 6, 7, 8}, find $A \cap (B \cup C)$

    (ii) If X = {*a, b, c, d*}, Y = {*c, d, e*} and Z = {*c, e, f, g*}

    prove that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

    (iii) If A = {2, 3, 4, 5}, B = {3, 5, 6, 9} and C = {2, 4, 6},

    prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

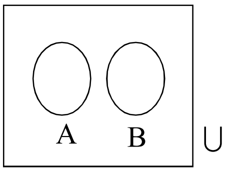
1.16 Application of set theory:

By number of elements of a set A, we mean the number of distinct elements of the set and we denote it by n(A). If n(A) is a natural number, then A is a non-empty finite set. For example, if A = {1, 2, 7, 9, 10}, then n(A) = 5.

Theorem 4 : If A and B are any two finite sets, then prove that

n($A \cup B$) = n(A) + n(B) – n ($A \cap B$).

Proof : Case I : Suppose that A and B are disjoint.



Then $A \cap B = \phi$ and so n ($A \cap B$) = 0

Let $n(A) = m_1$ and $n(B) = m_2$

$$\therefore\ n(AB) = m_1 + m_2$$

$$= m_1 + m_2 - 0$$

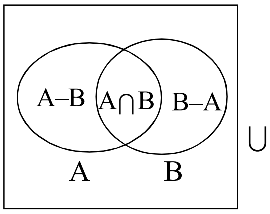
$$= n(A) + n(B) - n(A \cap B)$$

Case II : Suppose that A and B are not disjoint. Then $A \cap B \neq \phi$

We have,

$A = (A - B) \cup (A \cap B)$ where

$A - B$ and $A \cap B$ are disjoint.



$\therefore\ n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow\ n(A - B) = n(A) - n(A \cap B)$

Again we have,

$B = (B - A) \cup (A \cap B)$, where $B - A$ and $A \cap B$ are disjoint.

$\therefore\ n(B) = n(B - A) + n(A \cap B)$

$\Rightarrow n(B - A) = n(B) - n(A \cap B)$

Also, we have

$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$, where

$A - B$, $A \cap B$ and $B - A$ are pairwise disjoint.

$$\therefore\ n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$$

$$= n(A) - n(A \cap B) + n(A \cap B) + n(B) - n(A \cap B)$$

$$= n(A) + n(B) - n(A \cap B)$$

Theorem 5 : For any three finite sets A, B and C, prove that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Proof : We have,

$$n(A \cup B \cup C) = n[A \cup (B \cup C)]$$

$$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$$

$$= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup A \cap C)]$$

$$= n(A) + n(B) + n(C) - n(B \cap C)$$

$$- [n(A \cap B) + n(A \cap C) - n\{(A \cap B) \cap (A \cap C)\}]$$

$$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Example 24 : If X and Y are two sets such that $n(X) = 17$, $n(Y) = 23$ and $n(X \cup Y) = 38$, find $n(X \cap Y)$.

Solution : Given that $n(X) = 17$, $n(Y) = 23$ and $n(X \cup Y) = 38$

We have, $n(X \bigcup Y) = n(X) + n(Y) - n(X \bigcap Y)$

$$\Rightarrow 38 = 17 + 23 - n(X \bigcap Y)$$

$$\Rightarrow 38 = 40 - n(X \bigcap Y)$$

$$\Rightarrow (X \bigcap Y) = 40 - 38$$

$$= 2$$

Example 25 : If A and B are two sets such that $n(A \bigcup B) = 50$, $n(A) = 28$ and $n(B) = 32$, find $n(A \bigcap B)$.

Solution : Given that $n(A \bigcup B) = 50$, $n(A) = 28$ and $n(B) = 32$.

We have,

$$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B)$$

$$\Rightarrow 50 = 28 + 32 - n(A \bigcap B)$$

$$\Rightarrow 50 = 60 - n(A \bigcap B)$$

$$\Rightarrow n(A \bigcap B) = 60 - 50$$

$$= 10$$

Example 26 : If P and Q are two sets such that P has 40 elements, $P \bigcup Q$ has 60 elements and $P \bigcap Q$ has 10 elements, how many elements does Q have?

Solution : Given that $n(P) = 40$, $n(P \bigcup Q) = 60$ and $n(P \bigcap Q) = 10$.

We have, $n(P \bigcup Q) = n(P) + n(Q) - n(P \bigcap Q)$

$$\Rightarrow 60 = 40 + n(Q) - 10$$

$$\Rightarrow 60 = 30 + n(Q) \Rightarrow n(Q) = 60 - 30 = 30$$

Example 27 : Let A and B be two sets such that $n(A) = 20$, $n(A \bigcup B) = 42$ and $n(A \bigcap B) = 4$. Find (i) $n(B)$ (ii) $n(A - B)$ and $n(B - A)$.

Solution : Given that $n(A) = 20$, $n(A \bigcup B) = 42$ and $n(A \bigcap B) = 4$

(i) We have,

$$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B)$$

$$\Rightarrow 42 = 20 + n(B) - 4$$

$$\Rightarrow 42 = 16 + n(B)$$

$$\Rightarrow n(B) = 42 - 16$$

$$= 26$$

(ii) We have,

$$n(A–B) = n(A) - n(A \cap B)$$
$$= 20 - 4$$
$$= 16$$

(iii) We have,

$$n(B–A) = n(B) - n(A \cap B)$$
$$= 26 - 4$$
$$= 22$$

Example 28 : In a group of 400 people, 250 can speak Hindi and 200 can speak English. If each people can speak at least one of the languages, how many people can speak both Hindi and English?

Solution : Let A and B be the sets of people who can speak Hindi andn English respectively.

$\therefore\ n(A) = 250, n(B) = 200$

Since each people can speak at least one of the languages, therefore $n(AB) = 400$

We have $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$\Rightarrow 400 = 250 + 200 = n(A \cap B)$$
$$\Rightarrow n(A \cap B) = 450 - 400$$
$$= 50$$

$\therefore$ 50 people can speak both Hindi and English.

Example 29: In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many people like both coffee and tea?

Solution : Let A and B be the sets of people who like tea and coffee respectively.

$\therefore\ n(A) = 52, n(B) = 37$

Since each person likes at least one of the two drinks, therefore $n(A \cup B) = 70$

We have,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$\Rightarrow 70 = 52 + 37 - n(A \cap B)$$
$$\Rightarrow 70 = 89 - n(A \cap B)$$
$$\Rightarrow n(A \cap B) = 89 - 70$$
$$= 19$$

$\therefore$ 19 people like both tea and coffee.

Example 30: Every resident in Guwahati can speak at least one of the languages Assamese and English. If 70% can speak Assamese and 55% can speak English, what percent can speak both?

Solution : Let the total number of people living in Guwahati be 100. Also, let A and B be the sets of people who can speak Assamese and English respectively.

$\therefore$ $n(\text{A}) = 70$, $n(\text{B}) = 55$

Since every Guwahatian can speak at least one of the languages, therefore

$n(\text{A} \bigcup \text{B}) = 100$

We have $n(\text{A} \bigcup \text{B}) = n(\text{A}) + n(\text{B}) - n(\text{A} \bigcap \text{B})$

$$\Rightarrow 100 = 70 + 55 - n(\text{A} \bigcap \text{B})$$

$$\Rightarrow n(\text{A} \bigcap \text{B}) = 125 - 100$$

$$= 25$$

$\therefore$ 25% can speak both the languages.

Example 31: Out of 50 students, 35 drink tea and 40 drink coffee. What can be said about the number of students who drink both?

Solution : Let A and B be the sets of students who drink tea and coffee respectively

$\therefore$ $n(\text{A}) = 35$, $n(\text{B}) = 40$

We have

$n(\text{A} \bigcup \text{B}) \leq 50$ [$\because$ There may be some students who drink neither tea nor coffee]

$$\Rightarrow n(\text{A}) + n(\text{B}) - n(\text{A} \bigcap \text{B}) \leq 50$$

$$\Rightarrow 35 + 40 - n(\text{A} \bigcap \text{B}) \leq 50$$

$$\Rightarrow 75 - n(\text{A} \bigcap \text{B}) \leq 50$$

$$\Rightarrow 75 - 50 \leq n(\text{A} \bigcap \text{B})$$

$$\Rightarrow 25 \leq n(\text{A} \bigcap \text{B}) \quad \text{— (1)}$$

Again, $\text{AB} \subseteq \text{A}$, $\text{AB} \subseteq \text{B}$

$$\Rightarrow n(\text{A} \bigcap \text{B}) \leq n(\text{A}),\ n(\text{A} \bigcap \text{B}) \leq n(\text{B})$$

$$\Rightarrow n(\text{A} \bigcap \text{B}) \leq 35,\ n(\text{A}),\ n(\text{A} \bigcap \text{B}) \leq 40$$

$$\Rightarrow n(\text{A} \bigcap \text{B}) \leq 35 \quad \text{— (2)}$$

$\therefore$ (1) and (2) $\Rightarrow 25 \leq n(\text{A} \bigcap \text{B}) \leq 35$

Hence, the maximum and minimum number of students who drink both tea and coffee are 35 and 25 respectively.

Example 32: Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B. Is this data correct?

Solution : Let X and Y be the sets of car owners who owned car A and car B respectively.

$\therefore$ $n(X) = 400$, $n(Y) = 200$

and $n(X \cap Y) = 50$

We have,

$$\begin{aligned} n(X \cup Y) &= n(X) + n(Y) - n(X \cap Y) \\ &= 400 + 200 - 50 \\ &= 600 - 50 \\ &= 550 \end{aligned}$$

But $n(X \cup Y) \leq 500$

Hence, the given data is incorrect.

Example 33: In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi and all the people speak at least one of the two languages. How many people speak only English and not Hindi? How many people speak English?

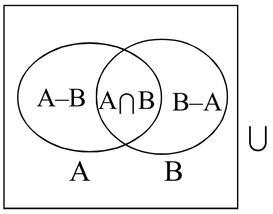
Solution : Let A and B be the sets of people who can speak Hindi and English respectively.

$\therefore$ $n(A) = 35$, $n(A \cap B) = 25$

Since all the people speak at least one of the languages, therefore $n(A \cup B) = 50$

We have,

$$\begin{aligned} n(B - A) &= n(A \cup B) - n(A) \\ &= 50 - 35 \\ &= 15 \end{aligned}$$



$\therefore$ 15 people can speak only English and not Hindi.

Again, we have

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow 50 = 35 + n(B) - 25$

$\Rightarrow 50 = 10 + n(B)$

$\Rightarrow n(B) = 40$

$\therefore$ The number of people who can speak English is 40.

Example 34: In a survey of 400 students in a school, 100 were listed as taking apple juice, 150 as taking orange juice and 75 were listed as taking both apple as well as orange juice. Find how many students were taking neither apple nor orange juice.

Solution : Let A and B be the sets of students who were listed as taking apple juice and orange juice respectively.

$\therefore$ $n(A) = 100$, $n(B) = 150$ and $n(AB) = 75$

We have,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 100 + 150 - 75$$
$$= 250 - 75$$
$$= 175$$

$\therefore$ The number of students who were listed as taking at least one of apple juice and orange juice is 175.

Hence, the number of students who were listed as taking neither apple juice nor orange juice is 400 – 175 = 225.

Example 35: In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find

(i) The number of people who read at least one of the newspapers.

(ii) The number of people who read none of the newspapers.

Solution : Let A, B and C be the sets of people who read the newspapers H, T and I respectively.

$\therefore$ $n(A) = 25$, $n(B) = 26$, $n(C) = 26$, $n(A \cap C) = 9$, $n(A \cap B) = 11$, $n(B \cap C) = 8$ and

$n(A \cap B \cap C) = 3$

(i) We have,

$$n(A \cup B \cup C = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - (C \cap A) + n(A \cap B \cap C)$$
$$= 25 + 26 + 26 - 11 - 8 - 9 + 3$$
$$= 77 + 3 - 28$$
$$= 80 - 28$$
$$= 52$$

$\therefore$ The number of people who read at least one of the newspapers is 52.

(ii) Again, the number of people who read none of the newspapers is 60 – 52 = 8.

Example 36: In a group of 65 people, 40 like cricket, 10 like both cricket and tennis and each people likes at least one of the games. How many likes tennis only and not cricket? How many like tennis?

Solution : Let A and B be the sets of people who like cricket and tennis respectively.

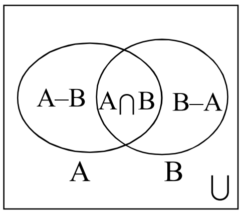
$\therefore$ $n(A) = 40$, $n(A \cap B) = 10$

Since each people likes at least one of the games, therefore $n(A \cup B) = 65$.

We have,

$n(B - A) = n(AB) - n(A)$
$= 65 - 40$
$= 25$

$\therefore$ 25 people like tennis only and not cricket.



Again, we have

$$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B)$$

$$\Rightarrow 65 = 40 + n(B) - 10$$

$$\Rightarrow 65 = 30 + n(B)$$

$$\Rightarrow n(B) = 65 - 30$$

$$= 35$$

$\therefore$ 35 people like tennis.

Example 37: A market research group conducted a survey of 2000 consumers and reported that 1720 consumers liked product P_1 and 1450 consumers liked product P_2. What is the least number that must have liked both the products?

Solution : Let A and B be the sets of consumers who liked products P_1 and P_2 respectively.

$\therefore$ $n(A) = 1720$, $n(B) = 1450$

We have, $n(A \bigcup B) \leq 2000$

$$\Rightarrow n(A) + n(B) - n(A \bigcap B) \leq 2000$$

$$\Rightarrow 1720 + 1450 - n(A \bigcap B) \leq 2000$$

$$\Rightarrow 3170 - n(A \bigcap B) \leq 2000$$

$$\Rightarrow n(A \bigcap B) \geq 3170 - 2000$$

$$\Rightarrow n(A \bigcap B) \geq 1170$$

Thus, the least value of $n(A \bigcap B)$ is 1170.

Hence, the least number of consumers who liked both the products is 1170.

Example 38: In a survey of 700 students in a college, 180 were listed as drinking Limca, 275 as drinking Miranda and 95 were listed as both drinking Limca as well as Miranda. Find how many students were drinking neither Limca nor Miranda.

Solution : Let A and B be the sets of students who were listed as drinking Limca and Miranda respectively.

$\therefore$ $n(A) = 180$, $n(B) = 275$ and $n(A \bigcap B) = 95$

We have,

$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B)$
$= 180 + 275 - 95$
$= 455 - 95$

$= 360$

$\therefore$ The number of students who were listed as drinking at least one of the drinks is 360.

Hence, the number of students who were listed as drinking neither Limca nor Miranda is $700 - 360 = 340$.

Example 39: In a class of 50 students, 24 students have taken History and 16 have taken History but not Geography. If all the students have taken at least one of the subjects, find

(i) The number of students who have taken both History and Geography.

(ii) The number of students who have taken Geography but not History.

Solution : Let A and B be the sets of students who have taken History and Geography respectively.

$\therefore$ $n(A) = 24$ and

$n(A - B) = 16$

(i) We have,

$$n(A - B) = n(A) - n(A \cap B)$$

$$\Rightarrow 16 = 24 - n(A \cap B)$$

$$\Rightarrow n(A \cap B) = 24 - 16$$

$$= 8$$

$\therefore$ 8 students have taken both History and Geography.

(ii) Since all the students have taken at least one of the subjects, therefore $n(A \cup B) = 50$

We have,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow 50 = 24 + n(B) - 8$$

$$\Rightarrow 50 = 16 + n(B)$$

$$\Rightarrow n(B) = 50 - 16$$

$$= 34$$

$$\therefore \; n(B - A) = n(B) - n(A \cap B)$$

$$= 24 - 8$$

$$= 26$$

Hence, the number of students who have taken Geography but not History is 26.

Example 40: In a club of 100 members, 20 do not like to play football, 35 do not like to play cricket and 50 like to play both the games. Find how many members

(i) like to play football only,

(ii) like to play cricket only

and (iii) do not like to play any one of the games.

Solution : Let A and B be the sets of club members who like to play football and cricket respectively.

$\therefore$ $n(A') = 20$, $n(B') = 35$ and $(A \cap B) = 50$

Also, let $\bigcup$ be the set of all club members.

$\therefore\ n(\bigcup) = 100$

Now, $n(A') = 20 \Rightarrow n(\bigcup) - n(A) = 20$

$\Rightarrow 100 - n(A) = 20$

$\Rightarrow n(A) = 80$

$n(B') = 35 \Rightarrow n(\bigcup) - n(B) = 35$

$\Rightarrow 100 - n(B) = 35$

$\Rightarrow n(B) = 65$

(i) We have,

$n(A - B) = n(A) - n(A \bigcap B)$

$= 80 - 50$

$= 30$

$\therefore$ 30 members like to play football only.

(ii) We have,

$n(B - A) = n(B) - n(A \bigcap B)$

$= 65 - 50$

$= 15$

$\therefore$ 15 members like to play cricket only.

(iii) We have,

$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B)$

$= 80 + 65 - 50$

$= 80 + 15$

$= 95$

$\therefore$ The number of club members who like to play at least one of the games is 95.

Hence, the number of club members who do not like to play any one of the games is

$n(\bigcup) - n(A \bigcup B)$

$= 100 - 95$

$= 5$

Example 41: In a group of students, 100 students know Hindi, 50 know English and 25 know both. Each of the students knows either Hindi or English, How many students are there in the group?

Solution : Let A and B be the sets of students who know Hindi and English respectively.

$\therefore$ $n(A) = 100$, $n(B) = 50$ and $n(A \cap B) = 25$

We have,

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 100 + 50 - 25 \\ &= 150 - 25 \\ &= 125 \end{aligned}$$

So, the number of students who know at least one of the languages is 125.

Also, it is given that each of the students knows either Hindi or English.

Hence the number of students in the group is 125.

Exercise 1.3

1. In a group of 800 people, 550 can speak Hindi and 450 can speak English. If each people can speak at least one of the languages, how many can speak both Hindi and English?
2. If A and B are two sets and U is the universal set such that $n(\cup) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, find $n(A' \cap B')$.
3. If X and Y are two sets such that $X \cup Y$ has 18 elements, X has 8 elements and Y has 15 elements, how many elements does $X \cap Y$ have?
4. In a committee, 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. How many speak at least one of these two languages?
5. A survey shows that 63% Indian like tea and 76% like coffee. If x% of the Indian like both tea and coffee, find the value of x.
6. In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find,
   (i) How many drink tea and coffee both.
   (ii) How many drink coffee but not tea.
7. In a certain locality of Guwahati a survey is conducted among 1000 families. It reveals that 300 subscribe The Assam Tribune, 250 subscribe The Sentinel and 100 subscribe both. Find the number of families who do not subscribe to any of the two newspapers.
8. In a survey of 100 persons, it was found that 28 read magazine A, 30 read magazine B, 42 read magazine C, 8 read magazines A and B, 10 read magazines A and C, 5 read magazines B and C and 3 read all the three magazines. Find,
   (i) how many read none of the three magazines?
   (ii) how many read the magazine C only?

9. In a survey of 100 persons, it is found that 42 read India Today, 30 Sunday, 28 Outlook, 10 India Today and Sunday, 5 India Today and Outlook, 8 Sunday and Outlook and 3 read all the three magazines. How many read none of the three magazines?
10. In a city, 20% of the population travels by car, 50% travels by bus and 10% travels by both car and bus. Find the percentage of population who are travelling by car or bus.
11. If A and B are two sets such that $n(\text{A}) = 115$, $n(\text{B}) = 326$ and $n(\text{A–B}) = 47$, find $n(\text{A} \bigcup \text{B})$

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Relations and Functions

2.1 Introduction :

In this chapter, we shall study some basic definitions and examples of relations and functions. Some functions related to Business and Economics will also be discussed.

2.2 Definitions : If a pair of objects x and y occur in a specified order such that x is followed by y, then the pair is called an ordered pair, which is expressed in the form (x, y). Here x is called the first component and y is called the second component of the ordered pair. If x and y are different, then the ordered pairs (x, y) and (y, x) are also different.

Let A and B be any two non-empty sets. Then the set of all possible ordered pairs (x, y), where $x \in \text{A}$ and $y \in \text{B}$, is defined as the Cartesian product of A and B and it is denoted by $\text{A} \times \text{B}$.

$\therefore\ \text{A} \times \text{B} = \{(x, y) \mid x \in \text{A}, y \in \text{B}\}$

Similarly, $\text{B} \times \text{A} = \{(y, x) \mid x \in \text{A}, y \in \text{B}\}$

For example, let A = {1, 2} and B= $\{a, b, c\}$.

Then

$$\text{A} \times \text{B} = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$$

$$\text{B} \times \text{A} = \{(a,1),(b,1),(c,1),(a,2),(b,2),(c,2)\}$$

$$\text{A} \times \text{A} = \{(1,1),(1,2),(2,1),(2,2)\} \text{ and}$$

$$\text{B} \times \text{B} = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$$

Example 1: If $(x+3,5) = (6,2x+y)$, find x and y.

Solution : Given that,

$$(x+3,5) = (6,2x+y)$$

$$\Rightarrow x+3 = 6, 5 = 2x+y$$

$$\Rightarrow x = 6-3,\ y = 5-2y$$

$$\Rightarrow x = 3,\ y = 5-2\times 3$$

$$= 5 - 6$$

$$= -1$$

$$\therefore\ x = 3,\ y = -1$$

Example 2: Find a and b if $\left(\frac{a}{3}+1,\ b-\frac{2}{3}\right)=\left(\frac{5}{3},\frac{1}{3}\right)$

Solution : Given that,

$$\left(\frac{a}{3}+1,\ b-\frac{2}{3}\right)=\left(\frac{5}{3},\frac{1}{3}\right)$$

$$\Rightarrow \frac{a}{3}+1=\frac{5}{3},\ b-\frac{2}{3}=\frac{1}{3}$$

$$\Rightarrow \frac{a}{3}=\frac{5}{3}-1,\ b=\frac{1}{3}+\frac{2}{3}$$

$$\Rightarrow \frac{a}{3}=\frac{2}{3},\ b=\frac{3}{3}$$

$$\Rightarrow a=2,\ b=1$$

Example 3: If G = {7, 8} and H = {5, 4, 2}, find $\mathrm{G}\times\mathrm{H}$ and $\mathrm{H}\times\mathrm{G}$.

Solution : Here, G = {7, 8} and

H = {5, 4, 2}

$$\therefore\ \mathrm{G}\times\mathrm{H}=\{(7,5),(7,4),(7,2),(8,5),(8,4),(8,2)\}$$

and $\mathrm{H}\times\mathrm{G}=\{(5,7),(5,8),(4,7),(4,8),(2,7),(2,8)\}$

Example 4: If $\mathrm{A}=\{x\in\mathrm{N}\mid x\le 4\}$ and $\mathrm{B}=\{y\in\mathrm{N}\mid 3<y\le 5\}$, find $\mathrm{A}\times\mathrm{A}$, $\mathrm{A}\times\mathrm{B}$, $\mathrm{B}\times\mathrm{A}$ and $\mathrm{B}\times\mathrm{B}$

Solution : Here, $\mathrm{A}=\{x\in\mathrm{N}\mid x\le 4\}$

= {1, 2, 3, 4}

and $\mathrm{B}=\{y\in\mathrm{N}\mid 3<y\le 5\}$

= {4, 5}

$$\therefore\ \mathrm{A}\times\mathrm{A}=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$$

$$\mathrm{A}\times\mathrm{B}=\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5),(4,4),(4,5)\}$$

$$\mathrm{B}\times\mathrm{A}=\{(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(5,4)\}$$

$$\mathrm{B}\times\mathrm{B}=\{(4,4),(4,5),(5,4),(5,5)\}$$

Example 5: If Let A ={1, 2, 3}, B = {4} and C = {5}. Verify that

(i) $A \times (B \bigcup C) = (A \times B) \bigcup (A \times C)$

(ii) $A \times (B \bigcap C) = (A \times B) \bigcap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

Solution : Here, A ={1, 2, 3}, B = {4} and C = {5}

(i) We have,

$$B \bigcup C = \{4,5\}$$

$$\therefore\ A \times (B \bigcup C) = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$$

Again, $A \times B = \{(1,4),(2,4),(3,4)\}$

$$A \times C = \{(1,5),(2,5),(3,5)\}$$

$$\therefore\ (A \times B) \bigcup (A \times C) = \{(1,4),(2,4),(3,4),(1,5),(2,5),(3,5)\}$$

Hence $A \times (B \bigcup C) = (A \times B) \bigcup (A \times C)$

(ii) We have,

$$B \bigcap C = \phi$$

$$\therefore\ A \times (B \bigcap C) = \phi$$

Again, $A \times B = \{(1,4),(2,4),(3,4)\}$

$$A \times C = \{(1,5),(2,5),(3,5)\}$$

$$\therefore\ (A \times B) \bigcap (A \times C) = \phi$$

Hence $A \times (B \bigcap C) = (A \times B) \bigcap (A \times C)$

(iii) We have,

$$B - C = \{4\}$$

$$\therefore\ A \times (B - C) = \{(1,4),(2,4),(3,4)\}$$

Again $A \times B = \{(1,4),(2,4),(3,4)\}$

$$A \times C = \{(1,5),(2,5),(3,5)\}$$

$$\therefore\ (A \times B) - (A \times C) = \{(1,4),(2,4),(3,4)\}$$

Hence $A \times (B - C) = (A \times B) - (A \times C)$

Example 6 : Let A ={1, 2, 3}, $B = \{x \mid x \in N \text{ and } x \text{ is prime} < 5\}$. Find $A \times B$ and $B \times A$.

Solution : Here,

$$A = \{1, 2, 3\}$$

and $B = \{x \mid x \in N \text{ and } x \text{ is prime} < 5\}$

$= \{2, 3\}$

$\therefore\ A \times B = \{(1,2),(1,3),(2,2),(2,3),(3,2),(3,3)\}$

and $B \times A = \{(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

2.3 Relation : A relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product A × B. Thus R is a relation from A to B if and only if $R \subseteq A \times B$.

$\therefore\ R = \{(a,b) \mid a \in A,\ b \in B\}$

In symbol, we write $(a,b) \in R \Leftrightarrow aRb$.

The set $\{a \mid (a,b) \in R\}$ is called the domain of R.

The set $\{b \mid (a,b) \in R\}$ is called the range of R.

The whole set B is called the codomain of the relation R. In general, range $\subseteq$ codomain.

For example, let A = {1, 2, 3}, B = (*a, b, c*}, $R_1 = \{(1,b),(2,c),(1,a),(3,a)\}$ and $R_2 = \{(1,a),(2,b),(a,b)\}$. Then R_1 is a relation from A to B because $R_1 \subseteq A \times B$. But $(a,b) \in B \times B$ and so $R_2 \not\subseteq A \times B$. Hence R_2 is not a relation from A to B.

Again, if $R \subseteq A \times A$, then we say that R is a relation from A to A or on A. Thus R is a relation from A to A if and only if $R \subseteq A \times A$.

$\therefore R = \{(a,b) \mid a,b \in A\}$

If $R = \phi$ and $R = A \times A$, then R is said to be an empty and universal relation respectively.

Example 7 : Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by $R = \{(x,y) \mid y = x+1\}$. Find R. Also write down the domain, codomain and range of R.

Solution : Here, A = {1, 2, 3, 4, 5, 6}

Now, $R = \{(x,y) \mid y = x+1, x,y \in A\}$

$= \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

$\therefore$ Domain of R = {1, 2, 3, 4, 5}

Codomain of R = {1, 2, 3, 4, 5, 6} = A

and range of R = {2, 3, 4, 5, 6}

Example 8 : Let A = {–2, – 1, 0, 1, 2}. Find the relations defined as follows :

(i) $R_1 = \{(a,b) \mid a \text{ is the square of } b\}$

(ii) $R_2 = \{(a,b) \mid a \text{ is less than } b\}$

(iii) $R_3 = \{(a,b) \mid a \text{ is equal to } b\}$.

Solution : Here, A ={–2, – 1, 0, 1, 2}

(i) $\therefore\ \mathrm{R}_1 = \{(a,b)/a$ is the square of $b, a, b \in \mathrm{A}\}$

$= \{(1,-1),(0,0),(1,1)\}$

(ii) $\therefore\ \mathrm{R}_2 = \{(a,b)/a$ is less than $b, a, b \in \mathrm{A}\}$

$= \{(-2,-1),(-2,0),(-2,1),(-2,2),(-1,0),(-1,1),(-1,2),(0,1),(0,2),(1,2)\}$

(iii) $\therefore\ \mathrm{R}_3 = \{(a,b)/a$ is equal to $a, b \in \mathrm{A}\}$

$= \{(-2,-2),(-1,-1),(0,0),(1,1),(2,2)\}$

Example 9 : Given the sets A = {1, 2, 3, 4, 5} and B = {1, 3, 5}, write R as a set of ordered pairs, if the relation R from A to B is defined by '*x* is less than *y*'. State the domain and the range of the relation.

Solution : Here, A = {1, 2, 3, 4, 5} and B = {1, 3, 5}

Now, R = $\{(x, y) \mid x$ is less than $y,\ x \in \mathrm{A}, y \in \mathrm{B}\}$

$= \{(1,3),(1,5),(2,3),(2,5),(3,5),(4,5)\}$

$\therefore$ Domain of R = {1, 2, 3, 4}

and range of R = {3, 5}

Example 10: Let A be the set of first ten natural numbers and the relation R on A is defined as $x\mathrm{R}y \Leftrightarrow x + 2y = 10.$ Find R and the domain and rage of R.

Solution : Here, A ={1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Now, R = $\{(x, y) \mid x + 2y = 10,\ x, y \in \mathrm{A}\}$

$= \{(x, y) \mid x = 10 - 2y,\ x, y \in \mathrm{A}\}$

$= \{(8,1),(6,2),(4,3),(2,4)\}$

$\therefore$ Domain of R = {2, 4, 6, 8}

and range of R = {1, 2, 3, 4}

Example 11: Let R be a relation on the set N of natural numbers defined by $\mathrm{R} = \{(a,b) \mid a + 3b = 12\}.$ Find (i) R (ii) domain of R and (iii) range of R.

Solution : (i) We have,

$\mathrm{R} = \{(a,b) \mid a + 3b = 12,\ a, b \in \mathrm{N}\}$

$= \{(a,b) \mid a = 12 - 3b,\ a, b \in \mathrm{N}\}$

$= \{(9,1),(6,2),(3,3)\}$

$= \{(3,3),(6,2),(9,1)\}$

(ii) Domain of R = {3, 6, 9}

(iii) Range of R = {1, 2, 3}

Example 12: Let A = {1, 2, 3, ..., 14}. Define a relation R from A to A by R = $\{(x,y) \mid 3x - y = 0\}$. Write down its domain, codomain and range.

Solution : Here, A = {1, 2, 3, ..., 14}

$$\text{Now} \quad \text{R} = \{(x,y) \mid 3x - y = 0,\ x, y \in \text{A}\}$$

$$= \{(x,y) \mid y = 3x,\ \ x, y \in \text{A}\}$$

$$= \{(1,3),(2,6),(3,9),(4,12)\}$$

$\therefore$ Domain of R = {1, 2, 3, 4}

Codomain of R = {1, 2, 3, ..., 14} =A

Range of R = {3, 6, 9, 12}

Example 13: Determine the domain and range of the relation R defined by R = $\{(x, x+5) \mid x \in \{0,1,2,3,4,5\}\}$

Solution : We have,

$$\text{R} = \{(x, x+5) \mid x \in \{0,1,2,3,4,5\}\}$$

$$= \{(0,5),(1,6),(2,7),(3,8),(4,9),(5,10)\}$$

$\therefore$ Domain of R = {0, 1, 2, 3, 4, 5}

and range of R ={5, 6, 7, 8, 9, 10}

Example 14: Let A = {1, 2, 3, 4, 6} and R be the relation on A defined by $\{(a,b) \mid b$ is exactly divisible by $a, a, b \in \text{A}\}$

(i) Write R in roster form.

(ii) Find the domain of R.

(iii) Find the range of R.

Solution : Here, A = {1, 2, 3, 4, 6}

(i) We have,

$$\text{R} = \{(a,b) \mid b \text{ is exactly divisible by } a, a, b \in \text{A}\}$$

$$= \{(1,1),(1,2),(1,3),(1,4),(1,6),(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(6,6)\}$$

(ii) Domain of R = {1, 2, 3, 4, 6}

(iii) Range of R = {1, 2, 3, 4, 6}

Example 15: Let A be the set of first five natural numbers and let R be a relation on A defined as follows : $(x, y) \in \text{R} \Leftrightarrow x \le y$

Express R as a set of ordered pair. Find the domain and range of R.

Solution : Here, A = {1, 2, 3, 4, 5}

We have, $\mathrm{R} = \{(x, y) \mid x \le y,\ x, y \in \mathrm{A}\}$

$= \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),$

$(4,4),(4,5),(5,5)\}$

$\therefore$ Domain of R = {1, 2, 3, 4, 5}

and range of R = {1, 2, 3, 4, 5}

Exercise 2.1

1. Find the values of a and b, if $(3a-2, b+3) = (2a-1, 3)$.
2. If $(x+1, y-2) = (3,1)$, find the values of x and y.
3. If P = {a, b, c} and Q = {r}, find $\mathrm{P} \times \mathrm{Q}$ and $\mathrm{Q} \times \mathrm{P}$.
4. Let A = {1, 2, 3}, B = {3, 4} and C = {4, 5, 6}. Find

   (i) $\mathrm{A} \times (\mathrm{B} \cap \mathrm{C})$ (ii) $(\mathrm{A} \times \mathrm{B}) \cap (\mathrm{A} \times \mathrm{C})$

   (iii) $\mathrm{A} \times (\mathrm{B} \cup \mathrm{C})$ (iv) $(\mathrm{A} \times \mathrm{B}) \cup (\mathrm{A} \times \mathrm{C})$
5. If A = {2, 3}, B = (6, 8}, C = {1, 2) and D = {6, 9}, verify that

   (i) $(\mathrm{A} \times \mathrm{B}) \cap (\mathrm{C} \times \mathrm{D}) = (\mathrm{A} \cap \mathrm{C}) \times (\mathrm{B} \cap \mathrm{D})$

   (ii) $(\mathrm{A} \times \mathrm{B}) \cup (\mathrm{C} \times \mathrm{D}) \subseteq (\mathrm{A} \cup \mathrm{C}) \times (\mathrm{B} \cup \mathrm{D})$
6. If A = {1, 3, 5}, B = {x, y}, find

   (i) $\mathrm{A} \times \mathrm{B}$ (ii) $\mathrm{B} \times \mathrm{A}$ (iii) $\mathrm{A} \times \mathrm{A}$ and (iv) $\mathrm{B} \times \mathrm{B}$
7. If $\mathrm{A} \times \mathrm{B} = \{(a, x), (a, y), (b, x), (b, y)\}$, find A and B.
8. Let A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}, verify that

   (i) $\mathrm{A} \times (\mathrm{B} \cap \mathrm{C}) = (\mathrm{A} \times \mathrm{B}) \cap (\mathrm{A} \times \mathrm{C})$

   (ii) $\mathrm{A} \times \mathrm{C}$ is a subset of $\mathrm{B} \times \mathrm{D}$.
9. If A = {1, 2, 3} and B = {2, 4}, find $\mathrm{A} \times \mathrm{A}$, $\mathrm{B} \times \mathrm{B}$, $\mathrm{A} \times \mathrm{B}$, $\mathrm{B} \times \mathrm{A}$ and $(\mathrm{A} \times \mathrm{B}) \cap (\mathrm{B} \times \mathrm{A})$.
10. A relation R is defined from the set A = {2, 3, 4, 5} to a set B = {3, 6, 7, 10} as follows :

    $(x, y) \in \mathrm{R} \Leftrightarrow x$ divides y

    Express R as a set of ordered pairs and determine the domain and range of R.
11. If R is the relation "less than" from A = {1, 2, 3, 4, 5} to B = {1, 4, 5}, write down R as a set of ordered pairs.
12. Write the relation $\mathrm{R} = \{(x, x^3) \mid x$ is a prime number $< 10\}$ in roster form.

13. Let R be the relation on Z defined by

$$R = \{(a,b) \mid a,b \in Z, a-b \text{ is an integer}\}$$

Find the domain and range of R.

14. If A = {1, 2, 3}, B = {4, 5, 6}, which of the following are relations from A to B? Give reasons in support of your answer.

(i) $\{(1,6),(3,4),(5,2)\}$

(ii) $\{(1,5),(2,6),(3,4),(3,6)\}$

(iii) $\{(4,2),(4,3),(5,1)\}$

(iv) $A \times B$

15. Let A = {2, 3, 4, 5} and B = {3, 6, 7, 10}. A relation R from A to B is defined as

$$R = \{(x, y) \mid x \text{ is relatively prime to } y,\ x \in A, y \in B\}$$

Find R, domain of R and range of R.

16. Let R be a relation defined on the set N of all natural numbers by $(a,b) \in R \Leftrightarrow x + 2y = 8$. Find R, domain of R and the range of R.

17. Let A = {3, 5}, B = (7, 11) and R = $\{(a,b) \mid a-b \text{ is odd}, a \in A, b \in B\}$. Show that R is an empty relation from A to B.

18. Determine the domain and range of the following relation :

$$(a,b) \mid a \in N, a < 5, b = 4$$

19. Let A = {11, 12, 13}, B ={8, 10, 12} and a relation R from A to B is defined by

R = $\{(x, y) \mid y = x-3, x \in A, y \in B\}$. Find R, domain of R and range of R.

20. If the set A has p elements, B has q elements, what is the number of elements of $A \times B$.

21. Let A and B be two sets such that *n*(A) = 3 and *n*(B) = 2. If $(x,1), (y,2), (z,1)$ are in $A \times B$ write A and B.

2.4 Function : Let A and B be any two non-empty sets. If *f* is a relation from A to B, then $f \subseteq A \times B$. In every ordered pair of *f*, the second component is called the image of the first component.

A relation *f* from a set A to a set B is said to be a function if every element of the set A has one and only one image in the set B.

Alternatively, given two non-empty sets A and B (not necessarily distinct), if by relation $f \subseteq A \times B$, we can associate each member of A to one and only one member of B, then this relation *f* is called a function or a mapping from the set A to the set B.

The set A is called the domain and the set B is called the codomain of the function *f*. If *f* is a function from A to B and $(x,y) \in f$, then we write $f(x) = y$, where y is called the image of x under *f* and x is called the preimage of y under *f*. The set of images of the elements of the domain is called the range of *f*. It is clear that range $\subseteq$ codomain. The function *f* from A to B is denoted by $f : \mathrm{A} \to \mathrm{B}$.

For example, let A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and $f = \{(1,2),(2,4),(3,6),(4,8)\}$. It is clear that $f \subseteq \mathrm{A} \times \mathrm{B}$ and so *f* is a relation from A to B. Also every element of A is associated with one and only one element of B, that is, every element of A has one and only one image in the set B. So *f* is a function from A to B. The domain and the range of *f* are {1, 2, 3, 4} and {2, 4, 6, 8} respectively.

2.5 Definitions : A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R, it is called a real function.

2.6 Functional value : Let $f : \mathrm{A} \to \mathrm{B}$ be any given function. If a $\in$ A, then *f*(a) is obtained by putting a for x in the mathematical expression of $f(x)$. The $f(a)$ is called the functional value of *f* at $x = a$.

2.7 Some special functions :

(i) **Identity function :** Let R be the set of real numbers. The function $f : \mathrm{R} \to \mathrm{R}$ defined by $f(x) = x, \ \forall x \in \mathrm{R}$ is called the identity function of R. The domain and range of *f* are R each.

(ii) **Constant function :** Let R be the set of real numbers. The function $f : \mathrm{R} \to \mathrm{R}$ defined by $f(x) = k, \ \forall x \in \mathrm{R}$ is called a constant function, where $k \in \mathrm{R}$ is a constant. Here domain f is R and its range is {k}.

(iii) **Polynomial function :** Let R be the set of real numbers. A function $f : \mathrm{R} \to \mathrm{R}$ defined by $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n, \ \forall x \in \mathrm{R}$ is said to be a polynomial function, where n is a non-negative integer and $a_0, a_1, a_2, ..., a_n \in \mathrm{R}$.

For example, $f(x) = x^2 + 6x + 2, \ g(x) = x^6 + \sqrt{3x} + 2$ are polynomial functions. But $h(x) = x^{3/2} + 7x - 2$ is not a polynomial function.

(iv) **Rational function :** A function f defined in a domain is called a rational function if it is of the form $f(x) = \dfrac{g(x)}{\phi(x)}$, where $g(x)$ and $\phi(x)$ are polynomial functions of x defined in the domain where $\phi(x) \neq 0$.

For example, $f : \mathrm{R} - \{2\} \to \mathrm{R}$ defined by $f(x) = \dfrac{x^2 + 6}{x - 2}$ is a rational function.

(v) **Modulus function :** Let R be the set of real numbers. The function $f : \mathrm{R} \to \mathrm{R}$ defined by

$$f(x) = |x|, \forall x \in \mathrm{R}$$

$$= \begin{cases} x, \text{ if } x \geq 0 \\ -x, if\, x < 0 \end{cases}$$

is called the modulus function.

2.8 Algebra of real function :

(i) **Addition of two functions :** Let $f : \mathrm{A} \to \mathrm{R}$ and $g : \mathrm{A} \to \mathrm{R}$ be any two real functions, where $\mathrm{A} \subseteq \mathrm{R}$. Then $(f + g) : \mathrm{A} \to \mathrm{R}$ is defined by

$$(f + g)(x) = f(x) + g(x), \forall x \in \mathrm{A}$$

(ii) **Subtraction of functions :** Let $f : \mathrm{A} \to \mathrm{R}$ and $g : \mathrm{A} \to \mathrm{R}$ be any two real functions, where $\mathrm{A} \subseteq \mathrm{R}$. Then $(f - g) : \mathrm{A} \to \mathrm{R}$ is defined by

$$(f - g)(x) = f(x) - g(x), \forall x \in \mathrm{A} \; f(1) = 1 + 2 = 3$$

(iii) **Multiplication of a function by a scalar :** Let $f : \mathrm{A} \to \mathrm{R}$ be a real function, where $\mathrm{A} \subseteq \mathrm{R}$ and a be a scalar. Then $af : \mathrm{A} \to \mathrm{R}$ is defined by

$$(af \supset (x) = a\, f(x), \forall x \in \mathrm{X}$$

(iv) **Multiplication of two functions :** Let $f : \mathrm{A} \to \mathrm{R}$ and $g : \mathrm{A} \to \mathrm{R}$ be any two real functions, where $\mathrm{A} \subseteq \mathrm{R}$. Then $fg : \mathrm{A} \to \mathrm{R}$ is defibed by

$$(fg)(x) = f(x) g(x), \forall x \in \mathrm{A}$$

(v) **Division of two functions :** Let $f : \mathrm{A} \to \mathrm{R}$ and $g : \mathrm{A} \to \mathrm{R}$ be any two real functions, where $\mathrm{A} \subseteq \mathrm{R}$. Then $\dfrac{f}{g}$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in \mathrm{A}, \text{ where } g(x) \neq 0.$$

Example 16: If $f(x) = x^2$, find the values of $f(0), f(1)$ and $f(-1)$.

Solution : Here, $f(x) = x^2$

$$\therefore \; f(0) = 0^2 = 0$$

$$f(1)=1^2=1$$

and $f(-1)=(-1)^2=1$

Example 17: If $f(x)=\begin{cases} x+2, 1\le x\le 2 \\ 1-2x,\ 2<x\le 3\end{cases}$, find $f(1), f(1.5), f(2), f(2.5)$ and $f(3)$.

Solution : Here, $f(x)=\begin{cases} x+2, 1\le x\le 2 \\ 1-2x,\ 2<x\le 3\end{cases}$

$\therefore\ f(1)=1+2=3$

$f(1.5)=1.5+2=3.5$

$f(2)=2+2=4$

$f(2.5)=1-2\times 2.5=1-5=-4$

and $f(3)=1-2\times 3=1-6=-5$

Example 18:

(i) If $f(x)=3x+1$, find $f(-2), f(0)$ and $f(2)$.

(ii) If $f(x)=x^2-5|x|+2$, find $f(0), f(1)$ and $f(-1)$.

(iii) If $f(x)=\begin{cases} 2x-1,\ x\le 0 \\ x+1,\ 0<x<2 \\ 4-5x,\ x\ge 2\end{cases}$, find $f(0), f(-1), f(1), f(2)$ and $f(4)$.

Solution : (i) Here, $f(x)=3x+1$

$\therefore\ f(-2)=3\times(-2)+1=-6+1=-5$

$f(0)=3\times 0+1=0+1=1$

and $f(2)=3\times 2+1=6+1=7$

(ii) Here, $f(x)=x^2-5|x|+2$

$\therefore\ f(0)=0-5\times 0+2=2$

$$\begin{aligned} f(1)=1^2-5\times|1|+2 &= 1-5\times 1+2 \\ &= 3-5 \\ &= -2 \end{aligned}$$

and
$$\begin{aligned} f(-1)=(-1)^2-5|-1|+2 &= 1-5\times 1+2 \\ &= 3-5 \\ &= -2 \end{aligned}$$

(iii) Here, $f(x)=\begin{cases} 2x-1, \; x \le 0 \\ x+1, \; 0<x<2 \\ 4-5x, \; x \ge 2 \end{cases}$

$$\therefore \; f(0)=2\times 0-1=0-1=-1$$

$$f(-1)=2\times(-1)-1=-2-1=-3$$

$$f(1)=1+1=2$$

$$f(2)=4-5\times 2=4-10=-6$$

and $f(4)=4-5\times 4=4-20=-16$

Example 19: If $f(x)=x^2$, find $\dfrac{f(1.1)-f(1)}{1.1-1}$

Solution : Here, $f(x)=x^2$

$$\therefore \; \frac{f(1.1)-f(1)}{1.1-1}=\frac{(1.1)^2-1^2}{0.1}$$

$$=\frac{1.21-1}{0.1}$$

$$=\frac{0.21}{0.1}$$

$$=2.1$$

Example 20: Let $f,g:\mathrm{R}\to\mathrm{R}$ be defined by $f(x)=x+1$ and $g(x)=2x-3$. Find $f+g, f-g$ and $\dfrac{f}{g}$

Solution : Here, $f,g:\mathrm{R}\to\mathrm{R}$ is defined by

$$f(x)=x+1, \; \forall x\in \mathrm{R}$$

and $g(x)=2x-3, \; \forall x\in \mathrm{R}$

$$\therefore \; (f+g)(x)=f(x)+g(x), \; \forall x\in \mathrm{R}$$

$$=x+1-(2x-3), \; \forall x\in R$$

$$=3x-2, \; \forall x\in R$$

$$(f-g)(x)=f(x)-g(x), \; \forall x\in \mathrm{R}$$

$$=x+1-(2x-3), \; \forall x\in R$$

$$= x + 1 - 2x + 3,\ \forall x \in R$$

$$= -x + 4,\ \forall x \in R$$

And $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{2x-3},\ \forall x\left(\neq \frac{3}{2}\right) \in \mathrm{R}$

Example 21: If $f(x) = (x-a)^2 (x-b)^2$, find $f(a+b)$.

Solution : Here, $f(x) = (x-a)^2 (x-b)^2$

$$\therefore \quad f(a+b) = (a+b-a)^2 (a+b-b)^2$$

$$= b^2 a^2$$

$$= a^2 b^2$$

Example 22:

(i) If $y = f(x) = \dfrac{ax-b}{bx-a}$, then prove that $x = f(y)$.

(ii) If $f(x) = \dfrac{1}{1-x}$, show that $f\left[f\{f(x)\}\right] = x$.

Solution : (i) Here, $y = f(x) = \dfrac{ax-b}{bx-a},\ x \neq \dfrac{a}{b}$

$$\therefore \quad f(y) = \frac{ay-b}{by-a}$$

$$= \frac{a\left(\frac{ax-b}{bx-a}\right) - b}{b\left(\frac{ax-b}{bx-a}\right) - a}$$

$$= \frac{\frac{a(ax-b) - b(bx-a)}{bx-a}}{\frac{b(ax-b) - a(bx-a)}{bx-a}}$$

$$= \frac{a^2x - ab - b^2x + ab}{abx - b^2 - abx + a^2}$$

$$= \frac{\left(a^2 - b^2\right)x}{a^2 - b^2}$$

$$= x$$

(ii) Here, $f(x) = \frac{1}{1-x}, \; x \neq 1$

$$\therefore \; f\{f(x)\} = \frac{1}{1 - f(x)}$$

$$= \frac{1}{1 - \frac{1}{1-x}}$$

$$= \frac{1}{\frac{1-x-1}{1-x}}$$

$$= \frac{1-x}{-x}$$

$$= \frac{x-1}{x}$$

$$\therefore \; f[f\{f(x)\}] = \frac{1}{1 - \frac{x-1}{x}}$$

$$= \frac{1}{\frac{x-(x-1)}{x}}$$

$$= \frac{x}{x-x+1}$$

$$= x$$

Example 23: Find $f\{f(x)\}$ if (i) $f(x) = \frac{2x+1}{3+x}$ (ii) $f(x) = \frac{x+1}{x-1}$

Solution : (i) Here, $f(x) = \frac{2x+1}{3+x}, \; x \neq -3$

$$\therefore\ f\{f(x)\}=\frac{2f(x)+1}{3+f(x)}$$

$$=\frac{2\left(\frac{2x+1}{3+x}\right)+1}{3+\frac{2x+1}{3+x}}$$

$$=\frac{\frac{2(2x+1)+3+x}{3+x}}{\frac{3(3+x)+2x+1}{3+x}}$$

$$=\frac{4x+2+3+x}{9+3x+2x+1}$$

$$=\frac{5x+5}{5x+10}$$

$$=\frac{5(x+1)}{5(x+2)}$$

$$=\frac{x+1}{x+2}$$

(ii) Here, $f(x)=\frac{x+1}{x-1},\ x\neq 1$

$$\therefore\ f\{f(x)\}=\frac{f(x)+1}{f(x)-1}$$

$$=\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$$

$$=\frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-(x-1)}{x-1}}$$

$$= \frac{2x}{2}$$

$$= x$$

Example 24: If $f(x) = b\left(\frac{x-a}{b-a}\right) + a\left(\frac{x-b}{a-b}\right)$, then prove that $f(a) + f(b) = f(a+b)$.

Solution : Here,

$$f(x) = b\left(\frac{x-a}{b-a}\right) + a\left(\frac{x-b}{a-b}\right)$$

$$\therefore \quad f(a) = b\left(\frac{a-a}{b-a}\right) + a\left(\frac{a-b}{a-b}\right)$$

$$= 0 + a$$

$$= a$$

$$f(b) = b\left(\frac{b-a}{b-a}\right) + a\left(\frac{b-b}{a-b}\right)$$

$$= b + 0$$

$$= b$$

Again, $f(a+b) = b\left(\frac{a+b-a}{b-a}\right) + a\left(\frac{a+b-b}{a-b}\right)$

$$= \frac{b^2}{b-a} + \frac{a^2}{a-b}$$

$$= \frac{b^2}{b-a} - \frac{a^2}{b-a}$$

$$= \frac{b^2 - a^2}{b-a}$$

$$= \frac{(b+a)(b-a)}{b-a}$$

$$= b + a$$

Also, $f(a) + f(b) = a + b = b + a$

$\therefore \quad f(a) + f(b) = f(a+b)$

Example 25: If $f(x) = \frac{ax+b}{bx+a}$, then prove that $f(x)f\left(\frac{1}{x}\right) = 1$.

Solution : Here, $f(x) = \frac{ax+b}{bx+a}$

$$\therefore\ f\left(\frac{1}{x}\right) = \frac{a.\frac{1}{x}+b}{b.\frac{1}{x}+a}$$

$$= \frac{\frac{a+bx}{x}}{\frac{b+ax}{x}}$$

$$= \frac{bx+a}{ax+b}$$

$$\therefore\ f(x)f\left(\frac{1}{x}\right) = \left(\frac{ax+b}{bx+a}\right)\left(\frac{bx+a}{ax+b}\right)$$

$$= 1$$

2.9 Some functions in business and economics:

We are giving below some common functions in economics and business.

(i) **Demand function :** Demand function gives the relationship between demand and price. We know that increase in price of a commodity usualy leads to a decrease in demand. So demand is inversely related to the price. A demand function is written as $y = f(x)$, where y is the quantity demanded and x is the price. For example, a linear demand function is $y = a - bx$, where a and b are positive constants.

(ii) **Supply function :** Supply function gives the relationship between quantity supplied of a commodity and its price. We know that increase in price of a commodity usually leads to a increase in supply. So supply is directly proportional to the price. A supply function is written as $y = f(x)$, where y is the quantity supplied and x is the price. For example, a linear supply function is $y = a + bx$, where a and b are positive constants.

(iii) **Cost function :** Cost function gives the relationship between total cost and the quantity produced of a certain commodity. A cost function is written as $y = f(x)$, where y is the total cost and x is the number of quantity produced.

(iv) **Revenue function :** Revenue function gives the relationship between the revenue obtained and the quantity sold of a certain commodity. A revenue function is written as $y = f(x)$, where y is the total revenue earned and x is the number of quantity sold.

(v) **Profit function :** The profit function p(x) is given by $p(x)=R(x)-c(x),$ where R(x) is the total revenue earned from the sale of x units of a commodity and $c(x)$ is the total cost of x units of the same commodity.

Example 26: The total cost function $C(x)$ of producing x items is given by

$$C(x)=\begin{cases} 1000+5x, \text{ when } 0\le x\le 500 \\ 2000+4x, \text{when} \, 500<x\le 2000 \end{cases}$$

Find the cost of producing (i) 430 items and (ii) 1200 items.

Solution : Here,

$$C(x)=\begin{cases} 1000+5x, \text{ when } 0\le x\le 500 \\ 2000+4x, \text{when} \, 500<x\le 2000 \end{cases}$$

(i) We have,

$$\begin{aligned} \text{C (430)} &= 1000+5\times 430 \\ &=1000+2150 \\ &=3150 \end{aligned}$$

(ii) We have,

$$\begin{aligned} \text{C(1200)} &= 2000+4\times 1200 \\ &=2000+48000 \\ &=6800 \end{aligned}$$

Example 27 : The total revenue in Rupees received from the sale of x units of a product is given by $\text{R}(x)=3x^2+36x+5.$ Find the total revenue earned by selling 5 units of the product.

Solution : Here, the total revenue function is given by

$$\text{R}(x)=3x^2+36x+5$$

$$\begin{aligned} \therefore \ \text{R}(5) &= 3\times 5^2+36\times 5+5 \\ &=75+180+5 \\ &=260 \end{aligned}$$

$\therefore$ The total revenue earned by selling 5 units of the product is Rs. 260.

Example 28 : The total cost C(x) in Rupees, associated with the production of x units of an item is given by

$$C(x)=0.005x^3-0.02x^2+30x+5000.$$

Find the total cost for the production of 2 units.

Solution : Here,

$$\therefore \ C(2)=0.005\times 2^3-0.02\times 2^2+30\times 2+5000$$

$= 0.005 \times 8 - 0.02 \times 4 + 60 + 5000$

$= 0.04 - 0.08 + 5060$

$= -0.04 + 5060$

$= 5059.96$

$\therefore$ The total cost for the production of 2 units is Rs. 5060 (nearly).

Exercise 2.2

1. A function f is defined by $f(x) = 2x - 5$. Write down the values of $f(0), f(7)$ and $f(-3)$.
2. Which of the following relations are functions? Give reasons.
   (i) $\{(2,1),(5,1),(8,1),(11,1),(14,1),(17,1)\}$
   (ii) $\{(2,1),(4,2),(6,3),(8,4),(10,5),(12,6),(14,7)\}$
   (iii) $\{(1,3),(1,5),(2,5)\}$
3. If $f(x) = x + \frac{1}{x}$, then prove that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.
4. If $f(x) = x^3 - \frac{1}{x^3}$, then show that $f(x) + f\left(\frac{1}{x}\right) = 0$
5. If $f(x) = \frac{2x+1}{3x-2} = y$, then prove that $x = f(y)$.
6. If $f(x) = x^2$ and $g(x) = 2x + 1$, find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.
7. Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f+g)(x), (f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.
8. If $f(x) = 3 - 4x + x^2$, find $f(0), f(-1)$ and $f(4)$.
9. If $f(x) = \frac{3x^2 - 2x + 4}{x - 1}$, find $f(-2)$ and $f(0)$.
10. If $y = f(x) = \frac{5x-3}{3x-5}$, show that $f(y) = x$.

11. If $f(x)=2x^2+x,$ show that $\dfrac{f(a+b)-f(a)}{b}=4a+2b+1.$

12. If $f(x)=x^3-3|x|,$ then find $f(0), f(1)$ and $f(-1).$

13. A function f is defined as follows :

$$f(x)=\begin{cases} x+2, \text{when } 0\le x\le 1 \\ 4-1, \text{when } 1<x\ge 3 \\ 2, \quad \text{when } x>3 \end{cases}$$

Find $f\left(\dfrac{1}{2}\right), f(4)$ and $f(2).$

14. The total cost function C(x) of producing x items is given by

$$C(x)=\begin{cases} 0.5x+17000, \text{ if } 0<x\le 10000 \\ 5x+22000 \quad \text{if } 10000<x\le 20000 \end{cases}$$

Find the cost of producing (i) 500 items and (ii) 15000 items.

15. The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x)=13x^2+26x+15$$

Find the total revenue earned by selling 6 units of the product.

16. The total cost C(x) in Rupees associated with the production of x units of an item is given by

$$C(x)=0.007x^3-0.003x^2+15x+4000.$$

Find the total cost for the production of 3 units.

ANSWERS

Exercise 1.1

1. (i) Not a set (ii) set (iii) not a set (iv) not a set (v) set (vi) set (vii) set.
2. (i) {–3, 3} (ii) {C, 0, L, E, G} (iii) {AB, BC, AC} (iv) {3, 5, 7} (v) {0, 1, 2, 3, 4}
(vi) {1, 3, 5, 7} (vii) {b, c, d, f, g, h, j} (viii) {1, 2, 3, 4, 5, 6}
3. (i) $\{x| x$ is a letter in the English Alphabet$\}$
(ii) $\{x\,|\,x\in \text{N}$ and $x<12\}$ (iii) $\{x| x$ is a factor of 105 and $x\ne 1\}$
(iii) $\{x\,|\,x=5^n, n\in \text{N}$ and $1\le n\le 4\}$ (v) $\{x\,|\,x=2n, n\in \text{N}\}$
(iv) $\{x\,|\,x\in \text{N}$ and $x<7\}$ (vii) $\{x\,|\,x$ is an odd natural number and $x<10\}$

4. (i) A = $\{0, \pm 1, \pm 2, \pm 3\}$ (ii) B = $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\right\}$

   (iii) C = {m, i, s, p} (iv) D = {1, 2, 3, 6, 9, 18} (v) E = {3}

5. (i) Empty set (ii) Not empty (iii) Empty set (iv) Empty set (v) Empty set (vi) Empty set (vii) Not empty

6. (i) Empty set (ii) Singleton set (iii) Empty set (iv) Singleton set.

7. (i) Finite (ii) Finite (iii) Infinite (iv) Finite (v) Infinite (vi) Infinite (vii) Infinite (viii) Finite

8. A = B = E

9. (i) A = B (ii) A $\neq$ B

ANSWERS

Exercise 1.2

1. (i) $A \subseteq B$ (ii) $A \subseteq B$ (iii) $A \subseteq B$ (iv) $A \subseteq B$ (v) $A \subseteq B$ (vi) $A \subseteq B$ (vii) $A \subseteq B$

2. P(A) = $\{\phi, \{1), \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\},$

   $\{2,34\}, \{1,2,3,4\}\}$

3. $\phi. \{a\}$

4. $A \subseteq B, A \subseteq C, B \subseteq C, D \subseteq A, D \subseteq B, D \subseteq C$

5. (i) {1, 2, 3, 5, 6} (ii) {1, 2, 3, 4, 5}

6. (i) {1, 2, 3, 4, 5, 6, 7, 8} (ii) {1, 2, 3, 4, 5, 7, 8, 9, 10, 11}

   (iii) {4, 5, 6, 7, 8, 9, 10} (iv) {4, 5, 6, 7, 8, 10, 11, 12, 13, 14}

   (v) {1, 2,3, 4, 5, 6, 7, 8, 9, 10, 11} (vi) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}

   (vii) {4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14} (viii) {4, 5} (ix) ϕ (x) {4, 5}

7. (i) {7, 9, 11} (ii) {11, 13} (iii) ϕ (iv) {11} (v) ϕ (vi) {7, 9, 11} (vii) ϕ (viii) {7, 9, 11}

   (ix) {7, 9, 11} (x) {7, 9, 11, 15}

8. (i) {3, 6, 9, 15, 18, 21} (ii) {3, 9, 15, 18, 21} (iii) {3, 6, 9, 12, 18, 21} (iv) {4, 8, 16, 20}

   (v) {2, 4, 8, 10, 14, 16} (vi) {5, 10, 20} (vii) {20} (viii) {4, 8, 12, 16} (ix) {2, 6, 10, 14}

(x) $\{5, 10, 15\}$ (xi) $\{2, 4, 6, 8, 12, 14, 16\}$ (xii) $\{5, 15, 20\}$

10. (i) $\{2, 3, 7, 11, 13\}$

11. (i) $\{8\}$ (ii) $\{3, 5\}$ (iii) $\{1, 3, 5, 8\}$ (iv) $\{1, 4, 6, 7, 8, 9\}$ (v) $\{7, 8\}$

13. (i) $\{x \mid x \in \mathrm{R} \text{ and } 5 < x < 8\}$ (ii) $\{2\}$ (iii) $\mathrm{A} \bigcup \mathrm{B} = \{x \mid x \in \mathrm{R} \text{ and } 1 \le x \le 10\}$

$\mathrm{A} \bigcap \mathrm{B} = \{x \mid x \in \mathrm{R} \text{ and } 3 \le x \le 7\}$

(iv) $\{2\}$

14. Yes, $\mathrm{A} \bigcup \mathrm{B} = \mathrm{B}$

15. (i) False (ii) True (iii) False

16. (i) $\{3, 5\}$

Exercise 1.3

1. 200
2. 300
3. 5
4. 60
5. $39 \le x \le 63$
6. (i) 16 (ii) 20
7. 550
8. (i) 20 (ii) 30
9. 20
10. 60%
11. 373

Exercise 2.1

1. $a = 1,\ b = 0$
2. $x = 2,\ y = 3$
3. $\mathrm{P} \times \mathrm{Q} = \{(a,r),(b,r),(c,r)\}$ $\mathrm{Q} \times \mathrm{P} = \{(r,a),(r,b),(r,c)\}$
4. (i) $\{(1,4),(2,4),(3,4)\}$ (ii) $\{(1,4),(2,4),(3,4)\}$

   (iii) $\{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6)\}$

   (iv) $\{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6)\}$

6. (i) $\{(1,x),(1,y),(3,x),(3,y),(5,x),(5,y)\}$

   (ii) $\{(x,1),(x,3),(x,5),(y,1),(y,3),(y,5)\}$

   (iii) $\{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}$

(iv) $\{(x,x),(x,y),(y,x),(y,y)\}$

7. A = $\{a, b\}$, B = $\{x, y\}$

9. $A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

$B \times B = \{(2,2),(2,4),(4,2),(4,4)\}$

$A \times B = \{(1,2),(1,4),(2,2),(2,4),(3,2),(3,4)\}$

$B \times A = \{(2,1),(2,2),(2,3),(4,1),(4,2),(4,3)\}$

$(A \times B) \cap (B \times A) = \{(2,2)\}$

10. R = $\{(2,6),(2,10),(3,3),(3,6),(5,10\}$, Domain of R = {2, 3, 5}, Range of R = {3, 6, 10}

11. R = $\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5),(4,5)\}$

12. R = $\{(2,8),(3,27),(5,125),(7,343)\}$

13. Domain of R = Z, Range of R = Z

14. (i) Not a relation (ii) Relation (iii) Not a relation (iv) Relation

15. R = $\{(2,3),(2,7),(3,7),(10,10),(4,3),(4,7),(5,3),(5,6),(5,7)\}$

Domain of R = {2, 3, 4, 5}, Range of R = {3, 6, 7, 10}

16. R = $\{(2,6),(4,2),(6,1)\}$, Domain of R = {2, 4, 6}, Range of R = {1, 2, 6}

18. Domain = {1, 2, 3, 4}, Range = {4)

19. R = $\{(11,8),(13,10)\}$, Domain of R = {11, 13}, Range of R = {8, 10}

20. pq

21. A = $\{x, y, z\}$, B = {1, 2}

Exercise 2.2

1. – 5, 9, – 11

2. (i) Yes (ii) Yes (iii) No

6. $x^2 + 2x + 1;\ x^2 - 2x - 1;\ 2x^3 + x^2;\ \dfrac{x^2}{2x+1},\ x \neq -\dfrac{1}{2}$

7. $\sqrt{x} + x : \sqrt{x} - x;\ x^{3/2};\ x^{-1/2},\ x \neq 0$

8. 3, 8, 3

9. $-\frac{20}{3}, -4$

12. 0, –2, –4

13. $\frac{5}{2}$, 2, 2

14. (i) 17250 (ii) 97000

15. Rs. 639

16. Rs. 4045.20 (nearly)

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Unit-III : Indices and Logarithm

Indices

Introduction :

A knowledge of powers or indices is necessary to understand algebraic processes.

Indices are used to show how many times a number has been multiplied by itself.

Suppose we have

$$5 \times 5 \times 5 \times 5$$

We write this as '5 to the power 4' i.e. 5^4.

The number 4 is called the power or index or exponent and 5 is the base of 5^4.

Thus if '*a*' is a real number and '*m*' is a positive integer then the repeated multiplication

$a \times a \times a \times .....$ to *m* factors is called m^{th} power of '*a*' and is given by a^m

$\therefore\ a^m = a \times a \times ....$ to *m* factors

Laws of Indices :

(For all rational numbers)

(Exponents - Positive and negative integers, fractional, zero and negative numbers)

(1) $a^m \times a^n = a^{m+n}$ (where *m* and *n* are positive integers, base $a \neq 0,1$)

Proof: $a^m \times a^n = (a \times a \times ....$ to *m* factors$) \times (a \times a \times ....$to *n* factors)

$= a \times a \times ....$ to (*m* + *n*) factors

$\therefore\ a^m \times a^n = a^{m+n}$ (by definition)

This result is known as Fundamental Index Law.

Similarly if *m, n* and *p* are positive integers then

$a^m \times a^n \times a^p = a^{m+n+p}$

(2) $a^m \div a^n = a^{m-n}$ if $m > n$

Proof: $a^m \div a^n = \dfrac{a^m}{a^n}$

$$= \frac{a \times a \times ....\text{to } m \text{ factors}}{a \times a \times ....\text{to } n \text{ factors}}$$

$= a \times a \times ....$ to (*m* – *n*) factors

$\therefore\ a^m \div a^n = a^{m-n}$ (by definition)

Note : $\dfrac{a^m}{a^n} = \dfrac{a \times a \times ....\text{to } m \text{ factors}}{a \times a \times ....\text{to } n \text{ factors}}$ (if *n* > *m*)

$$= \frac{1}{a \times a \times \ldots . \text{to}\,(n-m)\,\text{factors}}$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$$

(3) $\left(a^m\right)^n = a^{mn}$

Proof: $\left(a^m\right)^n = a^m \times a^m \times \ldots .$ to n factors

$= a^{m+m+} \ldots .$ to n factors

$\therefore \left(a^m\right)^n = a^{mn}$

(4) $(ab)^n = a^n b^n$

Proof: $(ab)^n = ab \times ab \times \ldots .$ to n factors

$= (a \times a \times \ldots .$ to n factors$) \times (b \times b \times \ldots .$ to n factors$)$

$\therefore (ab)^n = a^n b^n$

(5) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Proof: $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \ldots .$ to n factors

$$= \frac{a \times a \times \ldots . \text{ to}\,n\,\text{factors}}{b \times b \times \ldots . \text{ to}\,n\,\text{factors}}$$

$\therefore \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Assuming fundamental law of indices to be true for other values of *m* and *n* :

(6) Meaning of a^0

If either m or n is zero then

$a^m \times a^0 = a^{m+0} = a^m \quad \left(\because a^0 = 1,\ a \neq 0\right)$

$\Rightarrow a^0 = \frac{a^m}{a^m}$

$\Rightarrow a^0 = 1$

(7) Meaning of a^{-n}

For *n* being a positive integer, $a \neq 0, 1$ then

$$a^n \times a^{-n} = a^{n-n}$$

$$\Rightarrow a^n \times a^{-n} = a^0 = 1$$

$$\Rightarrow a^{-n} = \frac{1}{a^n}$$

(8) Meaning of $a^{\frac{1}{n}}$ (*n* being a positive integer, $a \neq 0, 1$)

By applying fundamental law of index, we have

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{ to } n \text{ factors}$$

$$= a^{\frac{1}{n}+\frac{1}{n}} + \ldots \text{ to } n \text{ factors}$$

$$= a^{\frac{1}{n}.n}$$

$$= a^1$$

$$= a$$

$\therefore a^{\frac{1}{n}}$ is the n^{th} root of *a*

$$\therefore \; a^{\frac{1}{n}} = \sqrt[n]{a}$$

(9) Meaning of $a^{\frac{m}{n}}$

By applying fundamental index law, we have

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{ to } n \text{ factors}$$

$$= a^{\frac{m}{n}+\frac{m}{n}} + \ldots \text{ to } n \text{ factors}$$

$$= a^{\frac{m}{n}.n}$$

$$= a^m$$

$$\therefore \left(a^{\frac{m}{n}}\right)^n = a^m$$

$$\Rightarrow a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}}$$

$$\Rightarrow a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$\therefore\ a^{\frac{m}{n}}$ is the n^{th} root of the m^{th} power of 'a'.

Few more results :

(i) $a^m = k \Rightarrow a = k^{\frac{1}{m}}$

Proof : $a^m = k$

$$\Rightarrow \left(a^m\right)^{\frac{1}{m}} = k^{\frac{1}{m}}$$

$$\Rightarrow a = k^{\frac{1}{m}}$$

(ii) $a^m = b^n \Rightarrow a = b^{\frac{n}{m}}$ and $b = a^{\frac{m}{n}}$

Proof : $a^m = b^n$

$$\Rightarrow \left(a^m\right)^{\frac{1}{m}} = \left(b^n\right)^{\frac{1}{m}}$$

$$\Rightarrow a^{\frac{m}{m}} = b^{\frac{n}{m}}$$

$$\Rightarrow a = b^{\frac{n}{m}}$$

And $b^n = a^m$

$$\Rightarrow \left(b^n\right)^{\frac{1}{n}} = \left(a^m\right)^{\frac{1}{n}}$$

$$\Rightarrow b^{\frac{n}{n}} = a^{\frac{m}{n}}$$

$$\Rightarrow b = a^{\frac{m}{n}}$$

(iii) $a^{\frac{1}{m}} = k \Rightarrow a = k^m$

Proof : $a^{\frac{1}{m}} = k$

$$\Rightarrow \left(a^{\frac{1}{m}}\right)^m = k^m$$

$$\Rightarrow a^{\frac{m}{m}} = k^m$$

$$\Rightarrow a = k^m$$

(iv) $a^m = a^n \Rightarrow m = n$

(v) $a^m = b^m \Rightarrow a = b$

Examples:

$3^9 \times 3^5 = 3^{9+5} = 3^{14}$

$3^9 \div 3^5 = 3^{9-5} = 3^4$

$\left(3^9\right)^5 = 3^{9\times5} = 3^{45}$

$3^1 = 3$

$3^0 = 1$

$3^{-4} = \frac{1}{3^4}$

$\frac{1}{3^{-4}} = 3^4$

$\left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5}$

$\left(\frac{3}{4}\right)^{-5} = \frac{4^5}{3^5}$

$\sqrt{5} = 5^{1/2}$

$\sqrt[4]{3} = 3^{1/4}$

$\sqrt[n]{3} = 3^{\frac{1}{n}}$

Summary :
Rule

1) $a^m \times a^n = a^{m+n}$
2) $a^m \div a^n = a^{m-n}$
3) $(a^m)^n = a^{mn}$
4) $a^1 = a$
5) $a^0 = 1$
6) $a^{-m} = \frac{1}{a^m}$
7) $\frac{1}{a^{-m}} = a^m$
8) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
9) $\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}$
10) $\sqrt{a} = a^{1/2}$
11) $\sqrt[3]{a} = a^{1/3}$
12) $\sqrt[n]{a} = a^{\frac{1}{n}}$
13) If $a^m = b^m$ then $a = b$
14) If $a^m = a^n$ then $m = n$
15) If $a^n = m$ then $a = m^{\frac{1}{n}} = \sqrt[n]{m}$

Worked out Examples

Example (1)

Simplify

(i) $16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}}$

$= 2^{4\times\frac{1}{4}}$

$\therefore\ 16^{\frac{1}{4}} = 2$

(ii) $(243)^{3/5} = \left(\left(3^5\right)^{\frac{1}{5}}\right)^3$

$$= \left(3^{5\times\frac{1}{5}}\right)^3$$

$$= \left(3^1\right)^3$$

$$= 3^3$$

$\therefore\ (243)^{3/5} = 27$

(iii) $\left(\frac{81}{16}\right)^{-3/4} = \frac{1}{\left(\frac{81}{16}\right)^{3/4}}$

$$= \left(\frac{16}{81}\right)^{3/4}$$

$$= \left(\left(\frac{16}{81}\right)^{\frac{1}{4}}\right)^3$$

$$= \left(\left(\frac{2}{3}\right)^{4\times\frac{1}{4}}\right)^3$$

$$= \left(\left(\frac{2}{3}\right)^1\right)^3$$

$$= \left(\frac{2}{3}\right)^3$$

$\therefore\ \left(\frac{81}{16}\right)^{-3/4} = \frac{8}{27}$

(iv) $\left(16x^2\right)^{3/4} = 16^{\frac{3}{4}}\left(x^2\right)^{\frac{3}{4}}$

$$= \left(16^{\frac{1}{4}}\right)^3\left(x^{2\times\frac{3}{4}}\right)$$

$$= \left(2^{4\times\frac{1}{4}}\right)^3 \left(x^{\frac{3}{2}}\right)$$

$$= 2^3 . x^{\frac{3}{2}}$$

$$\therefore \left(16x^2\right)^{3/4} = 8x^{\frac{3}{2}}$$

(v) $\dfrac{2x^6}{4x^{-2}} = \dfrac{1}{2}x^{6+2}$

$$\therefore \frac{2x^6}{4x^{-2}} = \frac{1}{2}x^8$$

(vi) $\dfrac{x^2 \times x^5}{\left(x^3\right)^3} = \dfrac{x^{2+5}}{x^3 \times x^3 \times x^3}$

$$= \frac{x^7}{x^{3+3+3}}$$

$$= \frac{x^7}{x^9}$$

$$= \frac{1}{x^{9-7}}$$

$$\therefore \frac{x^2 \times x^5}{\left(x^3\right)^3} = \frac{1}{x^2}$$

(vii) $\dfrac{3^{n+1} + 3^n}{3^{n+3} - 3^{n+1}} = \dfrac{3^n.3 + 3^n}{3^n.3^3 - 3^n.3^1}$

$$= \frac{3^n(3+1)}{3^n\left(3^3 - 3\right)}$$

$$= \frac{4}{27-3}$$

$$= \frac{4}{24}$$

$$\therefore \frac{3^{3+1} + 3^n}{3^{n+3} - 3^{n+1}} = \frac{1}{6}$$

(viii) $\sqrt{\dfrac{x^8}{4x^6}} = \sqrt{\dfrac{x^8.x^{-6}}{4}}$

$$=\sqrt{\frac{x^2}{4}}$$

$$=\left(\frac{x^2}{4}\right)^{1/2}$$

$$=\frac{x^{2\times\frac{1}{2}}}{2^{2\times\frac{1}{2}}}$$

$$\therefore\ \sqrt{\frac{x^8}{4x^6}}=\frac{x}{2}$$

(ix) $\dfrac{a^{-2}b^2}{a^{-1}+b^{-1}}=\dfrac{b^2/a^2}{\frac{1}{a}+\frac{1}{b}}$

$$=\frac{b^2}{a^2\left(\frac{a+b}{ab}\right)}$$

$$=\frac{b^2\times ab}{a^2(a+b)}$$

$$=\frac{b^3}{a(a+b)}$$

$$\therefore\ \frac{a^{-2}b^2}{a^{-1}+b^{-1}}=\frac{b^3}{a^2+ab}$$

(2) **Simplify :**

(i) $\dfrac{2^{m+3}\times 3^{2m-n}\times 5^{m+n+3}\times 6^{n+1}}{6^{m+1}\times 10^{n+3}\times 15^{m}}$

Solution: $\dfrac{2^{m+3}.3^{2m-n}.5^{m+n+3}.6^{n+1}}{6^{m+1}.10^{n+3}.15^{m}}=\dfrac{2^m.2^3.3^{2m}.3^{-n}.5^m.5^n.5^2(2\times 3)^{n+1}}{(2\times 3)^{m+1}(2\times 5)^{n+3}(2\times 3)^m}$

$$= \frac{2^m.2^3.3^{2m}.3^{-n}.5^m.5^n.5^3.2^n.2^1 3^n.3^1}{2^m.2^1.3^m.3.2^n.5^n.2^3.5^3.3^m.5^m}$$

$$= 1$$

(ii) $\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

Solution: $\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)^2 \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)$$

$$= \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right)$$

$$= \frac{2e^x}{2} \times \frac{2e^{-x}}{2}$$

$$= e^x \times e^{-x}$$

$$= e^{x-x}$$

$$= e^0$$

$$= 1$$

(iii) $\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2}$

Solution: $\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2}$

$$= \left(x^a x^{-b}\right)^{a^2+ab+b^2} \left(x^b.x^{-c}\right)^{b^2+bc+c^2} \left(x^c.x^{-a}\right)^{c^2+ca+a^2}$$

$$= x^{(a-b)(a^2+ab+b^2)}.x^{(b-c)(b^2+bc+c^2)}.x^{(c-a)(c^2+ca+a^2)}$$

$$= x^{a^3-b^3}.x^{b^3-c^3}.x^{c^3-a^3}$$

$$= x^{a^3-b^3+b^3-c^3+c^3-a^3}$$

$$= x^0$$

$$= 1$$

(iv) $$\frac{1}{1+x^{b-a}+x^{c-a}}+\frac{1}{1+x^{a-b}+x^{c-b}}+\frac{1}{1+x^{a-c}+x^{b-c}}$$

$$=\frac{1}{1+\frac{x^b}{x^a}+\frac{x^c}{x^a}}+\frac{1}{1+\frac{x^a}{x^b}+\frac{x^c}{x^b}}+\frac{1}{1+\frac{x^a}{x^c}+\frac{x^b}{x^c}}$$

$$=\frac{x^a}{x^a+x^b+x^c}+\frac{x^b}{x^b+x^a+x^c}+\frac{x^c}{x^c+x^a+x^b}$$

$$=\frac{x^a+x^b+x^c}{x^a+x^b+x^c}$$

$$=1$$

(3) **Prove that**

(i) $$\left(\frac{x^m}{x^n}\right)^{m+n}\times\left(\frac{x^n}{x^l}\right)^{n+l}\times\left(\frac{x^l}{x^m}\right)^{l+m}=1$$

$$\text{L.H.S}=\left(\frac{x^m}{x^n}\right)^{m+n}\times\left(\frac{x^n}{x^l}\right)^{n+l}\times\left(\frac{x^l}{x^m}\right)^{l+m}$$

$$=x^{(m-n)(m+n)}.x^{(n-l)(n+l)}.x^{(l-m)(l+m)}$$

$$=x^{m^2-n^2}.x^{n^2-l^2}.x^{l^2-m^2}$$

$$=x^{m^2-n^2+n^2-l^2+l^2-m^2}$$

$$=x^0$$

$$=1$$

= R.H.S.

(ii) $$\sqrt[lm]{\frac{x^l}{x^m}}\times\sqrt[mn]{\frac{x^m}{x^n}}\times\sqrt[nl]{\frac{x^n}{x^l}}=1$$

$$\text{L.H.S}=\sqrt[lm]{\frac{x^l}{x^m}}\times\sqrt[mn]{\frac{x^m}{x^n}}\times\sqrt[nl]{\frac{x^n}{x^l}}$$

$$=\left(\frac{x^l}{x^m}\right)^{\frac{1}{lm}}.\left(\frac{x^m}{x^n}\right)^{\frac{1}{mn}}.\left(\frac{x^n}{x^l}\right)^{\frac{1}{nl}}$$

$$= x^{\frac{l-m}{lm}} . x^{\frac{m-n}{mn}} . x^{\frac{n-l}{nl}}$$

$$= x^{\frac{l-m}{lm}+\frac{m-n}{mn}+\frac{n-l}{nl}}$$

$$= x^{\frac{\mathrm{ln}-mn+ml-nl+mn-lm}{lmn}}$$

$$= x^{\frac{0}{lmn}}$$

$$= x^{0}$$

$$= 1$$

= R.H.S.

(iii) $x^{\frac{m^2-n^2}{m+n}} . x^{\frac{n^2-p^2}{n+p}} . x^{\frac{p^2-m^2}{p+m}} = 1$

$$\text{L.H.S} = x^{\frac{m^2-n^2}{m+n}} . x^{\frac{n^2-p^2}{n+p}} . x^{\frac{p^2-m^2}{p+m}}$$

$$= x^{\frac{(m+n)(m-n)}{m+n}} . x^{\frac{(n+p)(n-p)}{n+p}} . x^{\frac{(p+m)(p-m)}{p+m}} \Rightarrow 2^{2x} = \frac{96}{24} = 4$$

$$= x^{m-n} . x^{m-p} . x^{p-m}$$

$$= x^{m-n+n-p+p-m}$$

$$= x^{0}$$

$$= 1$$

= R.H.S.

(iv) $\dfrac{\left(p^2-\dfrac{1}{q^2}\right)^p \left(p-\dfrac{1}{q}\right)^{q-p}}{\left(q^2-\dfrac{1}{p^2}\right)^q \left(q+\dfrac{1}{p}\right)^{p-q}} = \left(\dfrac{p}{q}\right)^{p+q}$

$$\text{L.H.S} = \frac{\left(p^2-\dfrac{1}{q^2}\right)^p \left(p-\dfrac{1}{q}\right)^{q-p}}{\left(q^2-\dfrac{1}{p^2}\right)^q \left(q+\dfrac{1}{p}\right)^{p-q}}$$

$$=\frac{\left(p+\frac{1}{q}\right)^{p}\left(p-\frac{1}{q}\right)^{p}\left(p-\frac{1}{q}\right)^{q-p}}{\left(q+\frac{1}{p}\right)^{q}\left(q-\frac{1}{p}\right)^{q}\left(q+\frac{1}{p}\right)^{p-q}}$$

$$=\frac{\left(p+\frac{1}{q}\right)^{p}\left(p-\frac{1}{q}\right)^{p+q-p}}{\left(q-\frac{1}{p}\right)^{q}\left(q+\frac{1}{p}\right)^{p-q+q}}$$

$$=\frac{\left(p+\frac{1}{q}\right)^{p}\left(p-\frac{1}{q}\right)^{q}}{\left(q-\frac{1}{p}\right)^{q}\left(q+\frac{1}{p}\right)^{p}}$$

$$=\frac{\left(\frac{pq+1}{q}\right)^{p}\left(\frac{pq-1}{q}\right)^{q}}{\left(\frac{pq-1}{p}\right)^{q}\left(\frac{pq+1}{p}\right)^{p}}$$

$$=\frac{(pq+1)^{p}(pq-1)^{q}}{q^{p}\,q^{q}}\times\frac{p^{q}\,p^{p}}{(pq-1)^{q}(pq+1)^{p}}$$

$$=\frac{p^{p+q}}{q^{p+q}}$$

$$=\left(\frac{p}{q}\right)^{p+q}$$

= R.H.S.

(4) (i) If $a^{x}=b^{y}=c^{z}$ and $abc=1$

show that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$

Proof : Given $a^x = b^y = c^z$

Let $a^x = b^y = c^z = k$

$\Rightarrow a^x = k$

$\Rightarrow a = k^{\frac{1}{x}}$

Similarly

$b = k^{\frac{1}{y}}$

and $c = k^{\frac{1}{z}}$

Again given abc =1

$\Rightarrow k^{\frac{1}{x}}.k^{\frac{1}{y}}.k^{\frac{1}{z}} = k^0$

$\Rightarrow k^{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} = k^0$

$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

(ii) If $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$ and abc = 1 show that $x + y + z = 0$

Proof : Given $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$

Let $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k$

$\Rightarrow a^{\frac{1}{x}} = k$

$\Rightarrow a = k^x$

Similarly $b = k^y$

and $c = k^z$

Again given abc = 1

$\Rightarrow k^x.k^y.k^z = k^0$

$\Rightarrow k^{x+y+z} = k^0$

$\Rightarrow x + y + z = 0$

(iii) If $m = a^x$, $n = a^y$ and $a^2 = \left(m^y.n^x\right)^z$ show that xyz =1

Proof : Given $a^2 = \left(m^y.n^x\right)^z$

$$\Rightarrow a^2 = \left[\left(a^x\right)^y \left(a^y\right)^x\right]^z$$

$$\Rightarrow a^2 = \left[a^{xy}\ a^{xy}\right]^z$$

$$\Rightarrow a^2 = \left[a^{xy+xy}\right]^z$$

$$\Rightarrow a^2 = \left[a^{2xy}\right]^z$$

$$\Rightarrow a^2 = a^{2xyz}$$

$$\Rightarrow 2 = 2xyz$$

$$\Rightarrow 1 = xyz$$

$\therefore\ xyz = 1$

5 (i) If $2^x = 3^y = 6^{-z}$ show that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$

Proof : Given $2^x = 3^y = 6^{-z}$

Let $2^x = 3^y = 6^{-z} = k$

Then $2^x = k$

$\Rightarrow 2 = k^{\frac{1}{x}}$ — (i)

Similarly, $3 = k^{\frac{1}{y}}$ — (ii)

and $6 = k^{-\frac{1}{z}}$ — (iii)

Since $6 = 2 \times 3$

$\Rightarrow k^{-\frac{1}{z}} = k^{\frac{1}{x}} \times k^{\frac{1}{y}}$ (From (i), (ii), (iii))

$$\Rightarrow k^{-\frac{1}{z}} = k^{\frac{1}{x}+\frac{1}{y}}$$

$$\Rightarrow \frac{-1}{z} = \frac{1}{x}+\frac{1}{y}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$$

(ii) If $2^x = 3^y = 5^z = 30^p$ prove that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{p}$

Proof : Given $2^x = 3^y = 5^z = 30^p$

Let $2^x = 3^y = 5^z = 30^p = k$

Then $2^x = k$

$\Rightarrow 2 = k^{\frac{1}{x}}$ — (i)

Similarly, $3 = k^{\frac{1}{y}}$

$5 = k^{\frac{1}{z}}$

and $30 = k^{\frac{1}{p}}$

Since $30 = 2 \times 3 \times 5$

$\Rightarrow k^{\frac{1}{p}} = k^{\frac{1}{x}} \times k^{\frac{1}{y}} \times k^{\frac{1}{z}}$

$\Rightarrow k^{\frac{1}{p}} = k^{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$

$\Rightarrow \frac{1}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

(iii) If $2^a = 5^b = 20^c$ show that $ab = c\ (a + 2b)$

Proof : Given $2^a = 5^b = 20^c$

Let $2^a = 5^b = 20^c = k$

then $2^a = k$

$\Rightarrow 2 = k^{\frac{1}{a}}$ — (i)

Similarly, $5 = k^{\frac{1}{b}}$ — (ii)

and $20 = k^{\frac{1}{c}}$ — (iii)

Since $20 = 2 \times 2 \times 5$

$\Rightarrow k^{\frac{1}{c}} = k^{\frac{1}{a}} \times k^{\frac{1}{a}} \times k^{\frac{1}{b}}$

$\Rightarrow k^{\frac{1}{c}} = k^{\frac{1}{a}+\frac{1}{a}+\frac{1}{b}}$

$\Rightarrow \frac{1}{c} = \frac{1}{a} + \frac{1}{a} + \frac{1}{b}$

$\Rightarrow \frac{1}{c} = \frac{b+b+a}{ab}$

$$\Rightarrow \frac{1}{c} = \frac{(a+2b)}{ab}$$

$$\Rightarrow ab = c(a+2b)$$

(6) If $x^y = y^x$ then prove that $\left(\frac{x}{y}\right)^{\frac{x}{y}} = x^{\frac{x}{y}-1}$ and if $x = 2y$ then prove that $y = 2$

Solution: Given $x^y = y^x$

$$\Rightarrow y = x^{\frac{y}{x}} \quad \text{— (i)}$$

Now $$\left(\frac{x}{y}\right)^{\frac{x}{y}} = \frac{x^{\frac{x}{y}}}{y^{\frac{x}{y}}}$$

$$= \frac{x^{\frac{x}{y}}}{\left(x^{\frac{y}{x}}\right)^{\frac{x}{y}}} \quad \text{(from (i))}$$

$$= \frac{x^{\frac{x}{y}}}{x^1}$$

$$= x^{\frac{x}{y}}.x^{-1}$$

$$\therefore \left(\frac{x}{y}\right)^{\frac{x}{y}} = x^{\frac{x}{y}-1}$$

Again given $x^y = y^x$ and $x = 2y$

$$\Rightarrow (2y)^y = (y)^{2y}, \quad \text{putting } x = 2y$$

$$\Rightarrow (2y)^y = (y^2)^y$$

$$\Rightarrow 2y = y^2$$

$$\Rightarrow 2y - y^2 = 0$$

$$\Rightarrow y(2-y) = 0$$

Either $y = 0$ or $y = 2$
But $y = 0$ is impossible
Hence when $y = 2$ then $x = 2y$

$$\Rightarrow x = 2.(2)$$

$$\Rightarrow x = 4$$

$$\therefore\ x = 4 \text{ and } y = 2$$

(7) (i) If $x = 2^{\frac{1}{3}} + 2^{-\frac{1}{3}}$ prove that $2x^3 = 6x + 5$

Solution : Given $x = 2^{\frac{1}{3}} + 2^{-\frac{1}{3}}$

Cubing both sides,

$$x^3 = \left(2^{\frac{1}{3}} + 2^{-\frac{1}{3}}\right)^3$$

$$\Rightarrow x^3 = \left(2^{\frac{1}{3}} + 2^{-\frac{1}{3}}\right)^3 + 3.2^{\frac{1}{3}}.2^{-\frac{1}{3}}\left(2^{\frac{1}{3}} + 2^{-\frac{1}{3}}\right)$$

$$\Rightarrow x^3 = 2 + 2^{-1} + 3.2^0.x$$

$$\Rightarrow x^3 = 2 + \frac{1}{2} + 3x$$

$$\Rightarrow x^3 = \frac{4 + 1 + 6x}{2}$$

$$\Rightarrow x^3 = \frac{6x + 5}{2}$$

$$\Rightarrow 2x^3 = 6x + 5$$

(ii) If $x = 5 - 5^{2/3} - 5^{1/3}$ prove that $x^3 - 15x^2 + 60x - 20 = 0$

Solution: Given $x = 5 - 5^{2/3} - 5^{1/3}$

$$\Rightarrow x - 5 = -\left(5^{2/3} - 5^{1/3}\right)$$

Cubing $(x - 5)^3 = -\left(5^{2/3} + 5^{1/3}\right)^3$

$$\Rightarrow -\left(x^3 - 15x^2 + 75x - 125\right) = \left(5^{2/3}\right)^3 + \left(5^{\frac{1}{3}}\right)^3 + 3.5^{\frac{2}{3}}.5^{\frac{1}{3}}\left(5^{\frac{2}{3}} + 5^{\frac{1}{3}}\right)$$

$$= 5^2 + 5 + 3 \times 5(-x+5)$$

$$= 25 + 5 + 15(-x+5)$$

$$\Rightarrow -x^3 + 15x^2 - 75x + 125 = 30 - 15x + 75$$

$$\Rightarrow -x^3 + 15x^2 - 60x + 125 - 105 = 0$$

$$\Rightarrow x^3 - 15x^2 + 60x - 20 = 0$$

Hence proved.

(8) **Solve :**

(i) $4^{x+2} + 2^{2x+3} = 96$

Solution: Given equation is

$$4^{x+2} + 2^{2x+3} = 96$$

$$\Rightarrow 2^{2(x+2)} + 2^{2x+3} = 96$$

$$\Rightarrow 2^{2x}.2^4 + 2^{2x}.2^3 = 96$$

$$\Rightarrow 2^{2x}\left(2^4 + 2^3\right) = 96$$

$$\Rightarrow 2^{2x}(16+8) = 96$$

$$\Rightarrow 2^{2x}(24) = 96$$

$$\Rightarrow 2^{2x} = \frac{96}{24} = 4$$

$$\Rightarrow 2^{2x} = 2^2$$

$$\Rightarrow 2x = 2$$

$$\Rightarrow x = 1$$

(ii) $9.81^x = \frac{1}{27^{x-3}}$

Solution: $9.81^x = \frac{1}{27^{x-3}}$

$$\Rightarrow 3^2.3^{4x} = \frac{1}{27^{x-3}}$$

$$\Rightarrow 3^2.3^{4x} = \frac{1}{3^{3(x-3)}}$$

$$\Rightarrow 3^{2+4x} = 3^{-3(x-3)}$$

$\Rightarrow 2+4x=-3(x-3)$

$\Rightarrow 2+4x=-3x+9$

$\Rightarrow 7x=7$

$\Rightarrow x=1$

(iii) Solve for x and y

$x^y = y^2$ and $y^{2y} = x^4$

Solution: Given $x^y = y^2$

$\Rightarrow \left(x^y\right)^y = \left(y^2\right)^y$

$\Rightarrow x^{y^2} = y^{2y}$

$\Rightarrow x^{y^2} = x^4$

$\therefore\ y^2 = 4$

$\Rightarrow y^2 - 4 = 0$

$\Rightarrow (y+2)\ (y-2) = 0$

Either $y = 2$ or $y = -2$

When $y = 2$

then $x^2 = 4$

$\Rightarrow x = \pm 2$

When $y = -2$ then

$x^{-2} = 4$

$\Rightarrow x = \pm\frac{1}{2}$

Exercise

(1) **Simplify**

(i) $8^{-2/3}$ $\left(\text{Ans } \frac{1}{4}\right)$

(ii) $(64)^{\frac{-5}{6}}\sqrt[6]{(27)^4}$ $\left(\text{Ans } \frac{9}{32}\right)$

(iii) $\dfrac{9\left(4^n\right)^2}{16^{n+1}-2^{n+1}.8^n}$ $\left(\text{Ans } \frac{9}{14}\right)$

(iv) $4^{\frac{1}{3}} \times \left(2^{\frac{1}{3}} \times 3^{\frac{1}{2}}\right)^{7} \div 9^{\frac{1}{4}}$ $\left(\text{Ans } 216\right)$

(v) $\dfrac{\left(3^{2n} - 5 \times 3^{2n-2}\right)\left(5 - 3 \times 5^{n-2}\right)}{5^{n-4}\left(9^{n+3} - 3^{2n}\right)}$ $\left(\text{Ans } \dfrac{275}{819}\right)$

(2) **Prove that**

(i) $\left(\dfrac{x^{b}}{x^{c}}\right)^{b+c-a}\left(\dfrac{x^{c}}{x^{a}}\right)^{c+a-b}\left(\dfrac{x^{a}}{x^{b}}\right)^{a+b-c} = 1$

(ii) $\dfrac{1}{1+x^{b-a}} + \dfrac{1}{1+x^{a-b}} = 1$

(iii) $\left(x^{\frac{1}{l-m}}\right)^{\frac{1}{l-n}} \times \left(x^{\frac{1}{m-n}}\right)^{\frac{1}{m-l}} \times \left(x^{\frac{1}{n-l}}\right)^{\frac{1}{n-m}} = 1$

(iv) $\dfrac{1}{1+x^{a-b}+x^{a-c}} + \dfrac{1}{1+x^{b-c}+x^{b-a}} + \dfrac{1}{1+x^{c-a}+x^{c-b}} = 1$

(v) $\left(\dfrac{x^{\frac{b}{c}}}{x^{\frac{c}{b}}}\right)^{\frac{1}{bc}} \times \left(\dfrac{x^{\frac{c}{a}}}{x^{\frac{a}{c}}}\right)^{\frac{1}{ca}} \times \left(\dfrac{x^{\frac{a}{b}}}{x^{\frac{b}{a}}}\right)^{\frac{1}{ab}} = 1$

(3) (i) If $a^{x} = b^{y} = c^{z}$ and $b^{2} = ac$

prove that $\dfrac{1}{x} + \dfrac{1}{z} = \dfrac{2}{y}$

(ii) If $2^{x} = 3^{y} = 12^{z}$ show that $z(x+2y) = xy$

(iii) If $3^{x} = 5^{y} = (75)^{z}$ show that $xy = z(2x+y)$

(iv) If $x^{a} = y^{b} = (xy)^{c}$ show that $ab = c(a+b)$

(v) If $2^{x} = 3^{y} = 6^{z}$ show that $\dfrac{1}{z} = \dfrac{1}{x} + \dfrac{1}{y}$

(vi) If $x^{a} = y^{b} = z^{c}$ and $xyz = 1$

Prove that $ab + bc + ca = 0$

(vii) If $a = b^x, b = c^y,\ c = a^z$

Prove that $xyz = 1$

(viii) If $a = xy^{p-1},\ b = xy^{q-1},\ c = xy^{r-1}$

Prove that $a^{q-r}b^{r-p}c^{p-q} = 1$

(ix) If $a^x = b^y = c^z = d^w$ then

Prove that $xy(z+w) = zw(x+y)$

(x) If $(56)^a = (5.6)^b = 10^c$

Show that $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$

(4) (i) If $x = 2^{2/3} + 2^{1/3}$ prove that $x^3 - 6x - 6 = 0$

(ii) If $a = 2^{\frac{1}{3}} - 2^{-1/3}$ prove that $2a^3 + 6a = 3$

(iii) If $x = 2 + 2^{2/3} + 2^{1/3}$ prove that $x^3 - 6x^2 + 6x - 2 = 0$

(iv) If $a^{1/3} + b^{1/3} + c^{1/3} = 0$ prove that $(a+b+c)3 = 27abc$

(v) If $x = 1 + 3^{\frac{2}{3}} + 3^{\frac{1}{3}}$ prove that $x^3 - 3x^2 - 6x + 4 = 0$

(5) Solve

(i) $4^{2x+1} = 8^{x+3}$ (Ans: 7)

(ii) $2^{x+3} + 2^{x+1} = 320$ (Ans: 5)

(iii) $4^x - 3.2^{x+2} + 2^5 = 0$ (Ans: 2, 3)

(iv) $3^{2x-5} + 9^{x-2} = 4$ $\left(\text{Ans}: \dfrac{5}{2}\right)$

(v) $2^x.6^y = 24$ and $2^{2x}.3y = 48$ (Ans: $x = 2, y = 1$)

(vi) $4^{x+2} + 2^{2x+1} = 36$ $\left(\text{Ans}: \dfrac{1}{2}\right)$

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Logarithm

Introduction :

Logarithm is a very important mathematical tool which helps to simplify and handle large numbers.

Definition:

If $a^x = \mathrm{N}\,(a > 0,\ a \neq 1)$ then the index x is called the logarithm of N with respect to the base 'a'

and is written as $x = \log_a \mathrm{N}$

Here $a^x = \mathrm{N}$ is called Exponential Form

and $x = \log_a$ N is called Logarithmic Form

Example : $2^3 = 8$ is in Exponential Form

$3 = \log_2 8$ is in Logarithmic Form

Laws of Logarithms:

(1) $\log_a(m \times n) = \log_a m + \log_a n$ [Product Form)

Proof: Let $\log_a m = x$

$\Rightarrow a^x = m$ — (i)

and $\log_a n = y$

$\Rightarrow a^y = n$ — (ii)

Now (i) $\times$ (ii) $\Rightarrow a^x \times a^y = m \times n$

$\Rightarrow a^{x+y} = m \times n$

$\Rightarrow \log_a(m \times n) = x + y$

$\Rightarrow \log_a(m \times n) = \log_a m + \log_a n$

(2) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$ [Quotient Formula]

Proof: Let $\log_a m = x$

$\Rightarrow a^x = m$ — (i)

and Let $\log_a n = y$

$\Rightarrow a^y = n$ — (ii)

$$\frac{\text{(i)}}{\text{(ii)}} \Rightarrow \frac{a^x}{a^y} = \frac{m}{n}$$

$$\Rightarrow a^{x-y} = \frac{m}{n}$$

$$\Rightarrow \log_a\left(\frac{m}{n}\right) = x - y$$

$$\Rightarrow \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

(3) $\log_a m = n\log_a m$ (Power Formula)

Proof : Let $\log_a m = x$

$$\Rightarrow a^x = m$$

$$\Rightarrow \left(a^x\right)^n = m^n \text{ (Taking } n\text{th power both sides)}$$

$$\Rightarrow a^{nx} = m^n$$

$$\Rightarrow \log_a m^n = nx$$

$$\Rightarrow \log_a m^n = n\log_a m$$

(4) Logarithm of 1 with respect to any base is 0.

Proof : For all values of a

We have $a^0 = 1,\ a \neq 0$ as 0^0 is not defined

$$\therefore\ \log_a 1 = 0$$

(5) Logarithm of any number with respect to the same base is 1.

Proof : We know

$$a^1 = a$$

$$\Rightarrow \log_a a = 1$$

(6) If $\log_a x = \log_a y$ then $x = y$

Proof : Given $\log_a x = \log_a y$

Let $\log_a x = \log_a y = m$

$$\Rightarrow a^m = x \quad \text{— (i)}$$

and $a^m = y$ — (ii)

From (i) and (ii) $x = y$

(7) $\log_a m = \log_b m \times \log_a b$ (Change of base)

or $\log_b m = \dfrac{\log_a m}{\log_a b}$

Proof : Let $\log_a m = x$

$$\Rightarrow a^x = m \quad \text{— (i)}$$

and let $\log_b m = y$

$$\Rightarrow b^y = m \quad \text{— (ii)}$$

From (i) and (ii)

$$a^x = b^y$$

$$\Rightarrow b^y = a^x$$

$$\Rightarrow b = a^{\frac{x}{y}}$$

$$\therefore \frac{x}{y} = \log_a b$$

$$\Rightarrow x = y \log_a b$$

$$\Rightarrow \log_a m = \log_b x \times \log_a b$$

or $\log_b m = \dfrac{\log_a m}{\log_a b}$

Corollary :

$$\log_a a \times \log_b b = 1$$

Proof : From proof (7) we have

$$\log_a a \times \log_b m \times \log_a b$$

If $m = a$ then

$$\log_a a \times \log_b a \times \log_a b$$

$$\Rightarrow 1 = \log_b a \times \log_a b$$

$$\Rightarrow \log_b a \times \log_a b = 1$$

$\Rightarrow \log_b a = \dfrac{1}{\log_a b}$

and $\log_a b = \dfrac{1}{\log_b a}$

Worked Out Examples

Example 1 : Express the following in logarithmic form:

(i) $x^2 = 9$

Solution: $x^2 = 9$

$\Rightarrow \log_x 9 = 2$

(ii) $\left(\dfrac{1}{2}\right)^{1/3} = x$

Solution: $\left(\dfrac{1}{2}\right)^{1/3} = x$

$\Rightarrow \log_{1/2} x = \dfrac{1}{3}$

(iii) $\sqrt[3]{27} = 3$

Solution: $\sqrt[3]{27} = 3$

$\Rightarrow (27)^{1/3} = 3$

$\Rightarrow \log_{27} 3 = \dfrac{1}{3}$

(iv) $5^x = 125$

Solution: $5^x = 125$

$\Rightarrow \log_5 125 = x$

Example 2 : Find the value of

(i) $\log_5 125$

Solution: Let $\log_5 125 = x$

$\Rightarrow 5^x = 125$

$$\Rightarrow 5^x = 5^3$$

$$\Rightarrow x = 3$$

$$\therefore \log_5 125 = 3$$

(ii) $\log_{\sqrt{2}}{}^{16}$

Solution: Let $\log_{\sqrt{2}} 16 = x$

$$\Rightarrow \left(\sqrt{2}\right)^x = 16 = 2^4 = \left(\sqrt{2}\right)^8$$

$$\Rightarrow x = 8$$

$$\therefore \log_{\sqrt{2}} 16 = 8$$

(iii) $\log_{\sqrt{5}} (.008)$

Solution: Let $\log_{\sqrt{5}} (.008) = x$

$$\Rightarrow \left(\sqrt{5}\right)^x = .008 = \frac{8}{1000}$$

$$\Rightarrow \left(5^{\frac{1}{2}}\right)^x = \frac{1}{125} = \frac{1}{5^3}$$

$$\Rightarrow 5^{\frac{x}{2}} = 5^{-3}$$

$$\Rightarrow \frac{x}{2} = -3$$

$$\Rightarrow x = -6$$

$$\therefore \log_{\sqrt{5}} (.008) = -6$$

(iv) $\log_4 \left(\frac{1}{2}\right)$

Solution: Let $\log_4 \left(\frac{1}{2}\right) = x$

$$\Rightarrow 4^x = \frac{1}{2}$$

$$\Rightarrow \left(2^2\right)^x = 2^{-1}$$

$$\Rightarrow 2^{2x} = 2^{-1}$$

$\Rightarrow 2x = -1$

$\Rightarrow x = -\frac{1}{2}$

$\therefore\ \log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$

Example 3 : Find the logarithm of

(i) 144 to the base $2\sqrt{3}$

Solution: Let $\log_{2\sqrt{3}} 144 = x$

$\Rightarrow \left(2\sqrt{3}\right)^x = 144$

$\Rightarrow \left(\sqrt{2^2 \times 3}\right)^x = 144$

$\Rightarrow \left(\sqrt{12}\right)^x = 144$

$\Rightarrow \left(12^{\frac{1}{2}}\right)^x = 12^2$

$\Rightarrow 12^{\frac{x}{2}} = 12^2$

$\Rightarrow \frac{x}{2} = 2$

$\Rightarrow x = 4$

$\therefore\ \log_{2\sqrt{3}} 144 = 4$

(ii) 125 to the base $5\sqrt{5}$

Solution: $\log_{5\sqrt{5}} 125 = x$

$\Rightarrow \left(5\sqrt{5}\right)^x = 125$

$\Rightarrow \left(\sqrt{5^2 \times 5}\right)^x = 125$

$\Rightarrow \left(\sqrt{125}\right)^x = 125$

$\Rightarrow \left(\sqrt{125}\right)^{\frac{x}{2}} = \left(125\right)^1$

$$\Rightarrow \frac{x}{2} = 1$$

$$\Rightarrow x = 2$$

$$\therefore \log_{5\sqrt{5}} 125 = 2$$

Example 4 : The logarithm of a number to the base $\sqrt{2}$ is *a*. What is its logarithm to the base $2\sqrt{2}$?

Solution: Let the number be *x*

Given $\log_{\sqrt{2}} x = a$

$$\Rightarrow x = \left(\sqrt{2}\right)^a$$

According to question, we have to find $\log_{2\sqrt{2}} x$

Let $\log_{2\sqrt{2}} x = b$

$$\Rightarrow x = \left(2\sqrt{2}\right)^b$$

$$\Rightarrow \left(\sqrt{2}\right)^a = \left(2\sqrt{2}\right)^b$$

$$\Rightarrow \left(2^{\frac{1}{2}}\right)^a = \left(\sqrt{2^3}\right)^b$$

$$\Rightarrow 2^{\frac{a}{2}} = 2^{\frac{3b}{2}}$$

$$\Rightarrow \frac{a}{2} = \frac{3b}{2}$$

$$\Rightarrow a = 3b$$

$$\Rightarrow b = \frac{a}{3}$$

$$\therefore \log_{2\sqrt{2}} x = \frac{a}{3}$$

Example 5 : Find the base of the logarithm of

(i) 9 is 2

Solution: Let the base be *x*

$$\therefore \log_x 9 = 2$$

$$\Rightarrow x^2 = 9$$

$$\Rightarrow x^2 = 3^2$$

$$\therefore\ x = 3$$

(ii) 324 is 4

Solution: Let the base be x

$$\therefore\ \log_x 324 = 4$$

$$\Rightarrow x^4 = 324$$

$$\Rightarrow x^4 = \left(3^4\sqrt{2}\right)^4 = \left(3\sqrt{2}\right)^4$$

$$\Rightarrow x = 3\sqrt{2}$$

$$\therefore\ \log_{3\sqrt{2}} 324 = 4$$

(iii) $\sqrt{5}$ is $\left(-\frac{1}{6}\right)$

Solution: Let the base be x

$$\therefore\ \log_x \sqrt{5} = -\frac{1}{6}$$

$$\Rightarrow (x)^{-\frac{1}{6}} = \sqrt{5} = 5^{\frac{1}{2}}$$

$$\Rightarrow (x)^{-\frac{1}{6}} = \left(5^{-3}\right)^{-1/6}$$

$$\Rightarrow x = 5^{-3} = \frac{1}{125}$$

$$\therefore\ \log_{\frac{1}{125}} \sqrt{5} = -\frac{1}{6}$$

Example 6 : Evaluate $a^{\log_a{}^x}$

Solution: Let $a^{\log_a{}^x} = y$

$$\Rightarrow \log_a y = \log_a x \qquad \left[\begin{array}{l} a^x = \mathrm{N} \\ \Rightarrow \log_a \mathrm{N} = x \end{array}\right]$$

$$\Rightarrow y = x$$

$$\therefore\ a^{\log_a^x} = x$$

Example 7 : If $\log_{10} 2 = 0.30103$ and $\log_{10} 3 = 0.47712$ find

(i) $\log_{10} 0.0075$

Solution: $\log_{10} 0.0075 = \log_{10}\left(\frac{75}{10000}\right)$

$$= \log_{10} 75 - \log_{10}\left(10^4\right)$$

$$= \log_{10}\left(3\times 5^2\right) - 4\log_{10}{}^{10}$$

$$= \log_{10} 3 + 2\log_{10} 5 - 4\log_{10} 10$$

$$= \log_{10} 3 + 2\log_{10}\left(\frac{10}{2}\right) - 4\log_{10} 10$$

$$= \log_{10} 3 + 2\log_{10} 10 - 2\log_{10} 2 - 4\log_{10} 10$$

$$= \log_{10} 3 - 2\log_{10} 10 - 2\log_{10} 2$$

$$= 0.47712 - 2(0.30103) - 2$$

$$= 0.47712 - 0.60206 - 2$$

$$= 0.47712 - 2.60206$$

$$\therefore\ \log_{10} 0.0075 = -2.12494$$

(ii) $\log_8 25$

Solution: $\log_8 25 = \log_{10} 25\ \log_8 10$

$$= \log_{10} 5^2 \frac{1}{\log_{10} 8}$$

$$= 2\log_{10} 5. \frac{1}{\log_{10} 2^3}$$

$$= 2\ \log_{10}\left(\frac{10}{2}\right)\frac{1}{3\log_{10} 2}$$

$$= 2\left[\log_{10} 10 - \log_{10} 2\right]\frac{1}{3(0.30103)}$$

$$= 2\left[1 - 0.30103\right]\frac{1}{(0.90309)}$$

$$= 2\left[0.69897\right]\left[\frac{1}{0.90309}\right]$$

$$=\frac{1.39794}{0.90309}$$

$$\therefore\ \log_8 25 = 1.54795$$

(8) **Prove that**

(i) $\log_2[\log_{\sqrt{2}}(\log_{\sqrt{3}} 9)] = 2$

Solution:

$$\text{L.H.S} = \log_2[\log_{\sqrt{2}}(\log_{\sqrt{3}} 9)]$$

$$= \log_2\left[\log_{\sqrt{2}}\left(\log_{\sqrt{3}}(\sqrt{3})^4\right)\right]$$

$$= \log_2[\log_{\sqrt{2}}(4\log_{\sqrt{3}}\sqrt{3})]$$

$$= \log_2[\log_{\sqrt{2}} 4] \qquad \because \log_a a = 1$$

$$= \log_2\left[\log_{\sqrt{2}}(\sqrt{2})^4\right]$$

$$= \log_2\left[\log_{\sqrt{2}}(\sqrt{2})^4\right]$$

$$= \log_2(4.1)$$

$$= \log_2 2^2$$

$$= 2\log_2 2$$

$$= 2$$

$$= \text{R.H.S.}$$

(ii) $\log_2 \log_{\sqrt{2}} \log_3 81 = 2$

Solution:

$$\text{L.H.S} = \log_2 \log_{\sqrt{2}} \log_3 81$$

$$= \log_2 \log_{\sqrt{2}}(\log_3 3^4)$$

$$= \log_2 \log_{\sqrt{2}}(4\log_3 3)$$

$$= \log_2 \log_{\sqrt{2}} 4\times 1 \qquad \because \log_a a = 1$$

$$= \log_2 \log_{\sqrt{2}} \left(\sqrt{2}\right)^4$$

$$= \log_2 \ 4\log_{\sqrt{2}} \sqrt{2}$$

$$= \log_2 (4 \times 1)$$

$$= \log_2 2$$

$$= 2\log_2 2$$

$$= 2 \times 1$$

$$= 2$$

$$= \text{R.H.S.}$$

(9) **Simplify :**

(i) $\log\frac{b}{c} + \log\frac{c}{a} + \log\frac{a}{b}$

Solution:

$$\log\frac{b}{c} + \log\frac{c}{a} + \log\frac{a}{b}$$

$$= \log\left(\frac{b}{c} \times \frac{c}{a} \times \frac{a}{b}\right) \qquad [\because \log m + \log n = \log(m \times n)]$$

$$= \log 1$$

$$= 0 \qquad \therefore \ \log\frac{b}{c} + \log\frac{c}{a} + \log\frac{a}{b} = 0$$

(ii) $\log_a b \times \log_b c \times \log_c d \times \log_d a$

Solution:

$$\log_a b \times \log_b c \times \log_c d \times \log_d a$$

Let $\log_a b = p$

$$\Rightarrow a^p = b \quad \text{— (i)}$$

$$\log_b c = q$$

$$\log_b q = c \quad \text{— (ii)}$$

$$\log_c d = r$$

$$\Rightarrow c^r = d \quad \text{— (iii)}$$

and $\log_d a = s$

$\Rightarrow d^s = a$ — (iv)

From (iv) $a = d^s$

$= \left(c^r\right)^s$, from (iii)

$= c^{rs}$

$= \left(b^q\right)^{rs}$ from (ii)

$= b^{qrs}$

$\Rightarrow a^1 = \left(a^p\right)^{qrs}$ from (i)

$\therefore\ pqrs = 1$

$\Rightarrow \log_a b \times \log_b c \times \log_c d \times \log_d a = 1$

(iii) $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$

Solution:

$$7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$$

$$= 7[\log 16 - \log 15] + 5[\log 25 - \log 24] + 3[\log 81 - \log 80]$$

$$= 7\left[\log 2^4 - \log(3\times 5)\right] + 5\left[\log 5^2 - \log(3\times 2)^3\right] + 3\left[\log 3^4 - \log\left(2^4\times 5\right)\right]$$

$$= 7[4\log 2 - \log 3 - \log 5] + 5[2\log 5 - \log 3 - 3\log 2] + 3[4\log 3 - 4\log 2 - \log 5]$$

$$= 28\log 2 - 7\log 3 - 7\log 5 + 10\log 5 - 5\log 3 - 15\log 2 + 12\log 3 - 12\log 2 - 3\log 5$$

$$= (28 - 15 - 12)\log 2 + (12 - 7 - 5)\log 3 + (10 - 7 - 3)\log 5$$

$$= (28 - 27)\log 2 + (12 - 12)\log 3 + (10 - 10)\log 5$$

$$= \log 2 + 0\log 3 + 0\log 5$$

$$= \log 2 + 0 + 0$$

$$= \log 2$$

$$\therefore\ 7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} = \log 2$$

10. **Prove that**

(i) $2\log a + 2\log a^2 + 2\log a^3 + ... + 2\log a^n = n(n+1)\log a$

Solution:

$$\text{L.H.S} = 2\log a + 2\log a^2 + 2\log a^3 + \dots + 2\log a^n$$

$$= 2\left[\log a + \log a^2 + \log a^3 + \dots + \log a^n\right]$$

$$= 2\left[\log a + 2\log a + 3\log a + \dots + n\log a\right]$$

$$\because \log m^n = n\log m$$

$$= 2\left[1 + 2 + 3 + \dots + n\right]\log a$$

$$= 2\frac{n(n+1)}{2}\log a$$

$$= n(n+1)\log a$$

$$= \text{R.H.S.}$$

(ii) $\dfrac{1}{\log_a abcd} + \dfrac{1}{\log_b abcd} + \dfrac{1}{\log_c abcd} + \dfrac{1}{\log_d abcd} = 1$

Solution:

$$\text{L.H.S} = \frac{1}{\log_a abcd} + \frac{1}{\log_b abcd} + \frac{1}{\log_c abcd} + \frac{1}{\log_d abcd}$$

$$= \log_{abcd} a + \log_{abcd} b + \log_{abcd} c + \log_{abcd} d$$

$$= \log_{abcd}(a \times b \times c \times d)$$

$$= \log_{abcd} abcc \qquad \because \log_a a = 1$$

$$= 1 = \text{R.H.S.}$$

(iii) $\dfrac{\log_a x}{\log_{ab} x} = 1 + \log_a b$

Solution:

$$\text{L.H.S} = \frac{\log_a x}{\log_{ab} x}$$

$$= \log_a x\left(\log_x ab\right)$$

$$= \log_a x\left(\log_x a + \log_x b\right)$$

$$= \log_a x \log_x a + \log_a x \log_x b$$

$$= 1 + \log_a b$$

$$= \text{R.H.S.}$$

(iv) $\frac{1}{2}\log\left(1+\frac{2x}{y}+\frac{x^2}{y^2}\right)=\log\left(1+\frac{x}{y}\right)$

Solution:

$$\text{L.H.S} = \frac{1}{2}\log\left(1+\frac{2x}{y}+\frac{x^2}{y^2}\right)$$

$$=\frac{1}{2}\log\left(\frac{y^2+2xy+x^2}{y^2}\right)$$

$$=\frac{1}{2}\log\left(\frac{x+y}{y}\right)^2$$

$$=\frac{2}{2}\log\left(\frac{x+y}{y}\right)$$

$$=\log\left(1+\frac{x}{y}\right)$$

$$=\text{R.H.S.}$$

11. If $\log\frac{x+y}{2}=\frac{1}{2}(\log x+\log 2)$

Prove that $\frac{x}{y}+\frac{y}{x}=2$

Solution:

Given $\log\frac{x+y}{2}=\frac{1}{2}(\log x+\log 2)$

$$\Rightarrow \log\frac{x+y}{2}=\frac{1}{2}\log xy$$

$$\Rightarrow \log\frac{x+y}{2}=\log(xy)^{1/2}$$

$$\Rightarrow \frac{x+y}{2}=(xy)^{1/2}$$

Squaring both sides

$$\left(\frac{x+y}{2}\right)^2 = xy$$

$$\Rightarrow x^2 + 2xy + y^2 = 4xy$$

$$\Rightarrow x^2 + y^2 = 2xy$$

$$\Rightarrow \frac{x^2 + y^2}{xy} = 2$$

$$\Rightarrow \frac{x}{y} + \frac{y}{x} = 2$$

Hence proved.

12. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$ show that $abc = 1$

Solution:

Given $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$

Let $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k$

$$\frac{\log a}{b-c} = k$$

$$\Rightarrow \log a = k(b-c) \quad \text{— (i)}$$

Similarly $\log b = k\,(c-a)$ — (ii)

$\log c = k\,(a-b)$ — (iii)

Adding (i), (ii) & (iii)

$$\log a + \log b + \log c = k\,(a - c + c - a + a - b)$$

$$\Rightarrow \log abc = \text{k.0}$$

$$\Rightarrow \log abc = \log 1$$

$$\Rightarrow abc = 1$$

Hence proved.

(13) If $a^2 + b^2 = c^2$

show that $\frac{1}{\log_{b+c}{}^{a}} + \frac{1}{\log_{c-b}{}^{a}} = 2$

Solution: L.H.S $= \frac{1}{\log_{b+c}{}^{a}} + \frac{1}{\log_{c-b}{}^{a}}$

$$= \log_a (b+c) + \log_a (c-b)$$

$$= \log_a (b+c) + \log_a [-1(b-c)]$$

$$= \log_a (b+c) + \log_a (-1) + \log_a (b-c)$$

$$= \log_a (b+c) - 0 + \log_a (b-c)$$

$$= \log_a (b+c)(b-c)$$

$$= \log_a (b^2 - c^2)$$

$$= \log_a a^2$$

$$= 2\log_a a$$

$$= 2$$

$\therefore \frac{1}{\log_{b+c}{}^{a}} + \frac{1}{\log_{c-b}{}^{a}} = 2$ Hence proved.

(14) If $\log_a m = n$ show that $\log_{\frac{1}{a}} (m) = -n$

Solution: Given $\log_a m = n$

$$\Rightarrow a^n = m$$

$$\Rightarrow \frac{1}{a^n} = \frac{1}{m}$$

$$\Rightarrow \left(\frac{1}{a}\right)^n = \frac{1}{m}$$

$$\Rightarrow n = \log_{\frac{1}{a}}{}^{\left(\frac{1}{m}\right)}$$

$$\Rightarrow n = \log_{\frac{1}{a}}{}^{(1)} - \log_{\frac{1}{a}}{}^{(m)}$$

$$\Rightarrow n = 0 - \log_{\frac{1}{a}} (m)$$

$\Rightarrow \log_{\frac{1}{a}}(m) = -n$

Hence proved.

(15) If $a^2 + b^2 = 14ab$

Prove that $\log\frac{1}{4}\left[\frac{1}{4}(a+b)\right] = \frac{1}{2}(\log + \log b)$

Solution: Given $a^2 + b^2 = 14ab$

$\Rightarrow a^2 + b^2 + 2ab = 14ab + 2ab$ [adding $2ab$ both sides]

$\Rightarrow (a+b)^2 = 16ab$

Taking logarithm both sides

$\log(a+b)^2 = \log 16ab$

$\Rightarrow 2\log(a+b) = \log(4^2 ab) = \log 4^2 + \log a + \log b$

$\Rightarrow 2\log(a+b) - 2\log 4 = \log a + \log b$

$\Rightarrow 2[\log(a+b) - \log 4] = \log a + \log b$

$\Rightarrow \log\left(\frac{a+b}{4}\right) = \frac{1}{2}[\log a + \log b]$

Hence proved.

(16) If $a^x = b^y = c^z$ and $b^2 = ac$

Show that $y = \frac{2xz}{x+z}$ by using logarithmic method.

Solution: Given $a^x = b^y = c^z$

Let $a^x = b^y = c^z = k$

$a^x = k$

$\Rightarrow \log a^x = \log k$

$\Rightarrow x\log a = \log k$

$\Rightarrow \log a = \frac{\log k}{x}$ — (i)

Similarly, $\log b = \frac{\log k}{y}$ — (ii) and $\log c = \dfrac{\log k}{z}$ — (iii)

Also given $b^2 = ac$

Taking logarithm both sides we have

$$\log b^2 = \log ac$$

$$\Rightarrow 2\log b = \log a + \log c$$

$$\Rightarrow \frac{2}{y}\log k = \frac{\log k}{x} + \frac{\log k}{z} \quad \text{[From (i), (ii) \& (iii)]}$$

$$\Rightarrow \frac{2}{y}\log k = \left(\frac{1}{x} + \frac{1}{z}\right)\log k$$

$$\Rightarrow \frac{2}{y} = \frac{z+x}{xz}$$

$$\Rightarrow y = \frac{2xz}{x+z}$$

Hence proved.

(17) If $\log_{10} 2 = x$ show that $\log_8 25 = \dfrac{2}{3}\left(\dfrac{1}{x} - 2\right)$

Solution:

$$\text{L.H.S} = \log_8 25$$

$$= \log_8 5^2$$

$$= 2\log_8 5$$

$$= 2[\log_{10} 5 \times \log_8 10]$$

$$= 2\left[\log_{10}\left(\frac{10}{2}\right) \times \frac{1}{\log_{10} 8}\right]$$

$$= 2\left[\{\log_{10} 10 - \log_{10} 2\} \times \frac{1}{3\log_{10} 2}\right]$$

$$= 2\left[(1-x) \times \frac{1}{3x}\right]$$

$\therefore\ \log_8^{25} = \frac{2}{3}\left[\frac{1}{x} - 1\right]$ Hence proved.

(18) Solve $\log_x(8x-3) - \log_x 4 = 2$

Solution:

$$\log_x(8x-3) - \log_x 4 = 2$$

$$\Rightarrow \log_x\left(\frac{8x-3}{4}\right) = 2$$

$$\Rightarrow x^2 = \frac{8x-3}{4}$$

$$\Rightarrow 4x^2 = 8x - 3$$

$$\Rightarrow 4x^2 - 8x + 3 = 0$$

$$\Rightarrow 4x^2 - 6x - 2x + 3 = 0$$

$$\Rightarrow 2x(2x-3) - (2x-3) = 0$$

$$\Rightarrow (2x-3)(2x-1) = 0$$

Either $2x - 3 = 0$ or $2x - 1 = 0$

$$\Rightarrow x = \frac{3}{2} \qquad \Rightarrow x = \frac{1}{2}$$

$$\therefore\ x = \frac{3}{2} \text{ or } \frac{1}{2}$$

(19) Show that the value of $\log_{10} 2$ lies between $\frac{3}{10}$ and $\frac{4}{13}$

Solution: Since $10^3 < 2^{10}$

$\Rightarrow \log 10^3 < \log 2^{10}$ (Taking logarithm)

$$\Rightarrow 3\log 10 < 10 \log 2$$

$$\Rightarrow \frac{3}{10} < \log 2 \quad \text{— (i)}$$

Also $2^{13} < 10^4$

$$\Rightarrow \log 2^{13} < \log 10^4$$

$$\Rightarrow 13 \log 2 < 4\log 10$$

$$\Rightarrow \frac{13}{1}\log 2 < 4$$

$$\Rightarrow \log 2 < \frac{4}{13} \quad \text{— (ii)}$$

From (i) and (ii)

$$\frac{3}{10} < \log 2 < \frac{4}{13}$$

Hence proved.

Exercise

(1) Determine the value of each of the following logarithms

(i) $\log_9 27$ (ii) $\log\sqrt[3]{100}$ (iii) $\log_5 625$ (iv) $\log_{2\sqrt{2}} \frac{1}{64}$

(v) $\log_2(-8)$ (vi) $\log(0.001)$ (vii) $\log_{5\sqrt{5}} 125$ (viii) $\log_{2\sqrt{3}} 1728$

Answers :

(i) $\frac{3}{2}$ (ii) $\frac{2}{3}$ (iii) 4 (iv) -4 (v) -4 (vi) 3 (vii) 2 (viii) 6

(2) Determine the base if

(i) $\log 324 = 4$ (ii) $\log 9 = 2$ (iii) $\log 400 = 4$ (iv) $\log 0.001 = -3$

Answers :

(i) $3\sqrt{2}$ (ii) 3 (iii) $2\sqrt{5}$ (iv) 10

(3) (i) Prove that $\log(1+2+3) = \log 1 + \log 2 + \log 3$

(ii) If, *a, b, c* are three consecutive integers, prove that $\log(1+ac) = 2\log b$

(iii) If $\log_a b = 10$, $\log_{6a}(32b) = 5$ [Ans: a = 3

Find the value of *a* and *b* b = 3^{10}]

(iv) If $\log_{10} 8.75 = 0.9406$

Find the value of $\log_{10} 875$ [Ans: 2.9406]

(v) Find the value of

$$\frac{\log\sqrt{27} + \log 8 - \log\sqrt{1000}}{\log 1.2}$$ [Ans: $\frac{3}{2}$]

(vi) Find the value of

$\log_4\left[\log_{2\sqrt{3}}\left(\log_{\sqrt{2}} 64\right)\right]$ [Ans: $\frac{1}{2}$]

(vii) Evaluate :

$\log_2 \log_3 \log_2 512$ [Ans: 1]

(viii) If $\log 2 = 0.30103$, find the value of $\log 0.005$ [Ans: 4.69897]

(ix) If $p = \log_{2a} a,\ q = \log_{3a} 2a,\ r = \log_{4a} 3a$

Show that $pqr = 2qr - 1$

(x) If $\log_a 3 = 10,\ \log_{6a} 96 = 5$, find the value of a [Ans: 3]

(4) Prove that

(i) $\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1$

(ii) $\log\frac{11}{15} + \log\frac{490}{297} - 2\log\frac{7}{9} = \log 2$

(iii) $16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = \log 5$

(iv) $7\log\frac{10}{9} - 2\log\frac{25}{24} + 3\log\frac{81}{80} = \log 2$

(5) (i) If $x^2 + y^2 = 6xy$

Prove that $2\log(x+y) = \log x + \log y + 3\log 2$

(ii) If $x^2 + y^2 = 23xy$

Prove that $\log\frac{a+b}{5} = \frac{1}{2}(\log a + \log b)$

(iii) If $a^2 + b^2 = 14ab$

Prove that $\log\left[\frac{1}{\sqrt{3}}(a-b)\right] = \frac{1}{2}(2\log 2 + \log a + \log b)$

(iv) If $a^2 + 4b^2 = 12ab$

Prove that $\frac{1}{4}(a+2b) = \frac{1}{2}(\log a + \log b)$

(v) If $a^2+b^2=7ab$

Prove that $\log\left[\frac{1}{3}(a+b)\right]=\frac{1}{2}(\log a+\log b)$

(6) If $\frac{\log a}{b-c}=\frac{\log b}{c-a}=\frac{\log c}{a-b}$

Show that $a^a b^b c^c=1$

(7) (i) If $x=1+\log_a bc,\ y=1+\log_b ca,\ z=1+\log_c ab$

Prove that $xyz=xy+yz+zx$

(ii) If $a^{3-x}b^{5x}=a^{x+5}b^{3x}$

Show that $x\log\left(\frac{b}{a}\right)=\log a$

(iii) If $\log\left(\frac{a+b}{3}\right)=\frac{1}{2}\ (\log a+\log b)$

Prove that $\frac{a}{b}+\frac{b}{a}=7$

(iv) If $2\log_8 N=p,\ \log_2(2N)=q$ and $q-p=4$
find the value of N

(v) If $a^x=b^y=c^z$

Prove that $\log_a(bcd)=x\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right)$

(vi) Prove that

$\log_a(ab)+\log_b(ab)=\log_a(ab)\log_b(ab)$

Antilogarithm

Common Logarithms :

Logarithms calculated with respect to base 10 is called common logarithm. It is denoted by $\log_{10}^{x}$. Sometimes it is also written as $\log x$.

Characteristics and Mantissa :

Logarithms of a number consists of two parts — one integral part which is called characteristics and the other decimal part which is called mantissa.

For example :

For log 24.68

24 is the integral part and 68 is the decimal part. The integral part may be zero, positive or negative. The decimal part is less than one and is always positive.

There are two rules for determining characteristics.

(1) Logarithms of Positive Numbers greater than or equal to 1 i.e. $N \geq 1$

Then characteristics of log N is positive and is one less than the number of digits in the integral part of N.

Example :	N	Characteristics of log N
	7821	(4 – 1) = 3
	654	(3 – 1) = 2
	5.23	(1 – 1) = 0
	86	(2 – 1) = 1
	60.33	(2 – 1) = 1

Now,

Characteristic of log 7821 $\rightarrow$ Number of digits in integral - 1

$\rightarrow$ 4 – 1 = 3

(2) Logarithms of Positive Number N < 1.

The characteristics of log N is negative and is numerically one greater than the number of zeroes immediately after the decimal point N.

Example :	N	Characteristics of log N
	0.0248	$-(1+1) = -2 = \bar{2}$
	0.6234	$-(0+1) = -1 = \bar{1}$
	0.0068	$-(2+1) = -3 = \bar{3}$
	0.04560	$-(1+1) = -2 = \bar{2}$

Now,

Characteristic of log 0.0248 = – (number of zeros after the decimal point and before a significant number + 1)

$= -(1+1)$

$= -2$

$= \bar{2}$ [–2 is written as $\bar{2}$]

Determination of Mantissa:

The mantissa of the common logarithm of a number can be determined from a table called Logarithm table. In the table two digit figures ranging from 10 to 99 appears at the extreme left hand column. At the top most row contains the figures from 0 to 9. Below each of the one digit figure we observe nubers containing four digits which are termed as mantissa. Each mantissa having a decimal point preceeding it are being dropped. This forms the main part of the body. At the extreme right of the log table, there is a side table containing 9 columns ranging from 1, 2, ..., 9 which is called Mean-Difference table.

Calculation of Mantissa by using log table

Let us suppose we want to find mantissa for 6.738.

(1) First we ignore the decimal point of the given number.

(2) We search for the first two digits (i.e. 67) in the extreme left hand column of the log table.

(3) Then we see horizontally to the right to the column headed by 3 of the top-most row which given 8280.

(4) For the fourth digit 8 we look at the mean-difference table and locate the number headed by 8 and move on the same horizontal line of 67. The figure is 5.

(5) Then log 6738 has mantissa

$$\begin{array}{r} .8280 \\ +5 \\ \hline 0.8285 \end{array}$$

Thus log 6.738 = Characteristic + Mantissa

= 0 + 0.8285

= 0.8285

and .8285 is the mantissa.

Antilogarithm

If the logarithm of a number m (positive) is n then m is called antilog of n and is written as antilog n i.e. $\log m = n$

$\Rightarrow m = \text{Antilog}\,n$

Example : $\log 21.43 = 1.3310$

$\Rightarrow 21.43 = \text{Antilog}\,1.3310$

Calculation of Antilog

In order to calculate antilog of a number we have to take help of antilogarithm table whose arrangement is similar to that of the table of logarithms.

Suppose $\log n = 2.6451$

$\therefore\ n = \text{Antilog}\ 2.6451$

Here the characteristic of log *n* is 2 and mantissa is 0.6451. Now we are to find .64 from the first column of the antilog table. On the right of .64 and in the column headed by 5 we find that the number is 4406. The mean difference for the last digit 1 along the row .64 is 1. Now adding we get the number as 4406 + 1 = 4407. Thus against the mantissa .6451 we get 4407. The characteristics of 2.6451 is 2. The number whose logarithm is 2.6451 should have 3 (2 + 1) digits in the integral part.

$\therefore$ Antilog 2.6451 = 440.7

Similarly, Antilog 1.6451 = 44.07

Antilog 0.6451 = 4.407

Antilog of a negative number

The mantissa of a logarithm is always positive. The following procedure is followed to make a negative number as positive.

$$\begin{aligned}\text{Antilog}(-1.6781) &= \text{Antilog}(-1-.6781)\\ &= \text{Antilog}(-2+2-1.6781)\\ &= \text{Antilog}(-2+.3219)\\ &= \text{Antilog}\left(\bar{2}.3219\right)\end{aligned}$$

The mantissa is 0.3219. The entry in the row corresponding .32 under the digit 1 is 2094. The mean difference in the same row under 9 is 4. Thus adding we get 2094 + 4 = 2098.

Since characteristics is $\bar{2}$, so antilot (– 1.6781) = antilog $\left(\bar{2}.3219\right)$

= 0.02098

Worked Out Examples

(1) **Evaluate :**

(i) Antilog 2.6124

Solution: The mantissa is .6124. From the antilog table, the entry in the row corresponding to .61 under the digit 2 is 4093. The mean difference for the last digit 4 corresponding to .61 is 4. Adding we get 4093 + 4 = 4097. Also the characteristic is 2 so we should have 3 (2 + 1) digits in the integral part.

$\therefore$ Antilog 2.6124 = 409.7

(ii) Antilog $\left(\bar{1}.3666\right)$

Solution: The mantissa is .3666. From the antilog table we search for .36 in the extreme left hand column and see horizontally to the column headed by 6 and find that the figure is 2323. The mean difference in the same row under 6 is 3. Thus adding we get 2323 + 3 = 2326. Since characteristic is $\bar{1}$ so

Antilog $\left(\bar{1}.3666\right)$ = 0.2326

(2) **Simplify :**

(i) $\bar{2}.6348+3.8061-\bar{4}.7432$

Solution:

$$\bar{2}.6348+3.8061-\bar{4}.7432$$
$$=-2+.6348+3.8061-(-4+.7432)$$
$$=-2+.6348+3.8061+(-4-.7432)$$
$$=(.6248+3.8061+4)-(2.7432)$$
$$=8.4409-2.7432$$
$$=5.6977$$

(ii) $\bar{3}.625\times 4$

Solution:

$$\bar{3}.625\times 4$$
$$=(-3+.625)\times 4$$
$$=-12+2.500$$
$$=-12+2+.500$$
$$=-10+.500$$
$$=\overline{10}.500$$

(iii) $\overline{14}.3224\div 3$

Solution: First we raise the negative number so that it is divisible by 3.

Now $\overline{14}.3224\div 3=\left(\overline{15}+1.3224\right)\div 3$
$$=\bar{5}+0.4408$$
$$=\bar{5}.4408$$

(3) Evaluate using logarithm table.

(i) 1771×0.687

Solution:

Let $x=1771\times 0.687$

Taking log both sides, we have

$$\log x=\log(1771\times 0.687)$$
$$=\log(1771)+\log(0.687)$$
$$=3.2482+\bar{1}.8370$$
$$=3+.2482-1+.8370$$

$= 3.2482 + .8370 - 1$

$= 4.0852 - 1$

$\therefore\ \log x = 3.0852$

$x = \text{Antilog}(3.0852)$ [Characteristic is 3 so putting decimal point after 4 (3+1)

$\therefore\ x = 1217$ digits of the value]

(ii) $\dfrac{(33.0)(27.2)}{15.8}$

Solution:

Let $x = \dfrac{(33.0)(27.2)}{15.8}$

Taking log both sides

$$\log x = \log \frac{(33.0)(27.2)}{15.8}$$

$$= \log[(33.0)(27.2)] - \log(15.8)$$

$$= \log(33.0) + \log(27.2) - \log(15.8)$$

$= 5.68$

$$= 1.5185 + 1.4346 - 1.1987$$

$$= 1.7544$$

$\therefore x = \text{Antilog}(1.7544)$ [Characteristic is 1 so we put decimal point

$= 5.680 \times 10$ after 2 (1+1) digits of the value]

$= 56.80$

(iii) $(35.28)^{1/7}$

Solution: Let $x = (35.28)^{\frac{1}{7}}$

Taking log both sides

$$\log x = \log\ (35.28)^{\frac{1}{7}}$$

$$= \frac{1}{7}\log(35.28)$$

$$= \frac{1}{7}(1.5475)$$

$\log x = 0.2211$

$\therefore x = \text{Antilog}(0.2211)$ [Characteristic is 0, so we put decimal point after 1 (0+1)

$x = 1.663$ digit of the value]

(iv) $\sqrt[7]{\dfrac{1}{0.8176\times 13.64}}$

Solution:

Let $x=\sqrt[7]{\dfrac{1}{0.8176\times 13.64}}$

Taking log both sides

$$\log x = \log\left(\frac{1}{0.8176\times 13.64}\right)^{\frac{1}{7}}$$

$$=\frac{1}{7}\left[\log 1-\log(0.8176\times 13.64)\right]$$

$$=\frac{1}{7}\left[0-\log(0.8176)-\log(13.64)\right]$$

$$=\frac{1}{7}\left[-\bar{1}.9125-1.1348\right]$$

$$=\frac{1}{7}\left[-(-1+.9125)-1.1348\right]$$

$$=\frac{1}{7}\left[+1-0.9125-1-0.1348\right]$$

$$=\frac{1}{7}\left[-1.0473\right]$$

$$=-0.1496$$

$$=-1+1-0.1496$$

$$=\bar{1}.8504$$

$\therefore\ x=\text{Antilog}\left(\bar{1}.8504\right)$

$\therefore\ x=0.7086$

(4) If log 2 = 0.3010, log 3 = 0.4771 and log 7 = 0.8451
Find (i) log 21 (ii) log 0.6 (iii) log (.014)

Solution:

(i) $\log 21 = \log(3\times 7)$

$$=\log 3+\log 7$$

$$=0.4771+0.8451$$

$\therefore \log 21 = 1.3222$

(ii) $\log 0.6 = \log \frac{6}{10}$

$= \log 6 - \log 10$

$= \log(2 \times 3) - 1$

$= \log 2 + \log 3 - 1$

$= 0.3010 + 0.4771 - 1$

$= 0.7781 - 1$

$= -0.2219$

$= -1 + 1 - 0.2219$

$\therefore \log(0.6) = \bar{1}.7781$

(iii) $\log(.014) = \log\left(\frac{14}{1000}\right)$

$= \log 14 - \log 10^3$

$= \log(2 \times 7) - 3\log 10$

$= \log 2 + \log 7 - 3$

$= 0.3010 + 0.8451 - 3$

$= 1.1461 - 3$

$= -1.8539$

$= -2 + 2 - 1.8539$

$= \bar{2} + 0.1461$

$\therefore \log(0.014) = \bar{2}.1461$

(5) Find the number of digits in 7^{20}

Solution: Let $x = 7^{20}$

Taking log both sides

$\log x = \log 7^{20}$

$= 20 \log 7$

$= 20 \times 0.8451$

$\therefore \log x = 16.9020$

Characteristics is 16

We know when $n > 1$ then

Characteristics = number of digits –1

$\Rightarrow 16 \quad = n - 1$

$\Rightarrow n \quad = 17$

$\therefore$ Number of digits in 7^{20} is 17.

(6) Find the number of zeros between the decimal and first significant figure of $(0.04)^9$

Solution :

Let $x = (0.04)^9$

Taking log both sides we have

$$\log x = \log (0.04)^9$$
$$= 9 \log (0.04)$$
$$= 9 \log \left(\frac{4}{100}\right)$$
$$= 9 \left[\log 4 - \log 100\right]$$
$$= 9 \left[\log 2^2 - \log 10^2\right]$$
$$= 9 \left[2 \log 2 - 2 \log 10\right]$$
$$= 18 \left[\log 2 - \log 10\right]$$
$$= 18 \left[0.3010 - 1\right]$$
$$= 18 \left[-0.699\right]$$
$$= -12.582$$
$$= -13 + 13 - 12.582$$

$\therefore \log x = \overline{13}.0418$

Characteristics is 13

We know when $n < 1$ then

Characteristics = Number of zeros + 1

$\Rightarrow$ Char. $= n + 1$

$\Rightarrow$ n = Char. –1

$\Rightarrow$ n = 13 – 1 = 12

$\therefore$ Number of zeros between the decimal and first significant figure of $(0.04)^9$ is 12.

7. Solve the equation using logarithm table

(i) $3^{x-1} = 4.5^{1-3x}$

Solution :

$$3^{x-1} = 4.5^{1-3x}$$

Taking log both sides

$$\log\left(3^{x-1}\right) = \log\left(4.5^{1-3x}\right)$$

$$\Rightarrow (x-1)\log 3 = \log 4 + \log 5^{(1-3x)}$$

$$\Rightarrow x\log 3 - \log 3 = \log 4 + (1-3x)\log 5$$

$$\Rightarrow x\log 3 - \log 3 = \log 4 + \log 5 - 3x\log 5$$

$$\Rightarrow x\log 3 + 3x\log 5 = \log 3 + \log 5 + \log 4$$

$$\Rightarrow x[\log 3 + 3\log 5] = \log(3\times 5\times 4)$$

$$\Rightarrow x\left[\log 3 + \log 5^3\right] = \log 60$$

$$\Rightarrow x[\log 3 + \log 125] = \log 60$$

$$\Rightarrow x[\log(3\times 125)] = \log 60$$

$$\Rightarrow x\ \log 375 = \log 60$$

$$\Rightarrow x = \frac{\log 60}{\log 375}$$

$$\Rightarrow x = \frac{1.7782}{2.5740}$$

$$\therefore\ x = 0.6910$$

(ii) $6^{3-4x}.4^{x+5} = 8$

Solution : $6^{3-4x}.4^{x+5} = 8$

Taking log both sides, we have

$$\log\left(6^{3-4x}.4^{x+5}\right) = \log 8$$

$$\Rightarrow \log 6^{3-4x} + \log 4^{x+5} = \log 2^3$$

$$\Rightarrow (3-4x)\log 6 + (x+5)\log 4 = 3\log 2$$

$$\Rightarrow (3-4x)(0.7782) + (x+5)(0.6021) = 3(0.3010)$$

$$\Rightarrow 2.3346 - 3.1128x + 0.6021x + 3.0105 = 0.9030$$

$$\Rightarrow 5.3451 - 2.5107x = 0.9030$$

$$\Rightarrow -2.5107x = 0.9030 - 5.3450$$

$$\Rightarrow -2.5107x = -4.4419$$

$$\Rightarrow x = \frac{4.4419}{2.5107}$$

$$\therefore x = 1.77$$

Method of Interpolation

The method of Interpolation is based on the Principle of Proportional parts which states that for a small change in numbers, the corresponding change in their logarithms are proportional to the change in the numbers.

Given $\log 34.938 = 1.5432980$ and $\log 34.939 = 1.5433105$ find by interpolation $\log 349.387$.

Solution: First we find the value of log 34.9387. The number 34.9387 lies between 34.938 and 34.939.

Given $\log 34.939 = 1.5433105$

$\log 34.938 = 1.5432980$

Difference in logarithmic value = .0000125

Difference between the number .001

and difference between 34.9387 and 34.938 is .0007

Difference between numbers	Difference in log value
.001	.0000125
.0007	x

$$\therefore x = .0000125 \times \frac{.0007}{.001}$$

$$x = .00000875$$

$\therefore \log 34.938 = 1.5432980$

$.0007 = .00000875$

$\log 34.9387 = 1.54330675$

Hence $\log 349.387 = 2.54330675$

(i) Find N whose $\log N = 2.92479$

Solution: Given $\log N = 2.92479$

$$\Rightarrow N = \text{Antilog } 2.92479$$

Ignoring characteristic, Antilog 924 = 8395

Adding mean difference for 7 = 14

Adding mean difference for 9 = 1 | 7

$\therefore$ Antilog 92479 = 8411

Now the characteristics 2 indicates that the number must have 3 integral places, hence

N = Antilog 2.92479 = 841.1

(ii) Find N where log N = $\bar{1}.63345$

Solution: Given log N = $\bar{1}.63345$

$\Rightarrow$ N = Antilog $\bar{1}.63345$

Ignoring characteristic, Antilog 633 = 4295

Adding mean difference for 4 = 4

Adding mean difference for 5 = 5

$\therefore$ Antilog 63345 = 4304

Now the characteristic $\bar{1}$ indicates that there are no zeros after the decimal point

$\therefore$ N = Antilog $\bar{1}.63345 = 0.4304$

Exercise

(1) Find the characteristics of the following :

- (i) log 3486.4 Ans: 3
- (ii) log 40.32 Ans: 1
- (iii) log 0.11 Ans: $\bar{1}$
- (iv) log 0.076 Ans: $\bar{2}$
- (v) log 6.1 Ans: 0
- (vi) log 0.0091 Ans: $\bar{3}$

(2) Find the logarithmic value of

- (i) 6241 Ans: 3.7953
- (ii) 0.8861 Ans: $\bar{1}.9474$
- (iii) 3.2540 Ans: 0.5124
- (iv) 0.0032 Ans: $\bar{3}.5051$

(3) Find Antilog of the following numbers :

- (i) 1.3248 Ans: 21.13
- (ii) 0.1261 Ans: 1.334
- (iii) $\bar{1}.0583$ Ans: 0.1144
- (iv) –2.8254 Ans: 0.001495

(4) If Antilog 1.3621 = 23.02. Find Antilog of the following numbers.

- (i) 2.3621 Ans: 230.2

Antilogarithm

(ii) 0.3621 Ans: 2.302

(iii) $\bar{1}.3621$ Ans: 0.2302

(iv) $\bar{2}.3621$ Ans: 0.02302

(5) Find the values of the following using tables:

(i) $\dfrac{(33.0)(27.2)}{15.8}$ Ans : 56.80

(ii) $\dfrac{24.395 \times (3.16)^3}{8.79}$ Ans: 87.6

(iii) $\log\left\{\dfrac{(7.2)^3 \times (0.016)^4}{\left(\dfrac{6}{5}\right)^{15}}\right\}$ Ans: –5.7997

(6) Find the number of digits in the following numbers.

(i) 2^{30} Ans: 10

(ii) 4^{15} Ans: 10

(iii) 7^{20} Ans: 17

(iv) 5^{25} Ans: 18

(7) Find the number of zeros between the decimal point and the first significant number.

(i) $(12.4)^{-15}$ Ans: 16

(ii) $(0.0013)^{20}$ Ans: 57

(iii) $\left(\dfrac{1}{12}\right)^{100}$ Ans: 107

UNIT - IV

QUADRATIC EQUATION AND LINEAR SIMULTANEOUS EQUATIONS

4.1. **Introduction :** An algebraic equation is a mathematical statement in which two expression are equal to each other. If the equality is true for all values of the unknown quantities, the equation becomes an identity, some example of identities are

$$(x+2)(x-2)=x^2-4$$

$$(x+b)^2=x^2+2xb+b^2$$

$$(x-a)^3=x^3-3x^2a+3xa^2-a^3 \text{ etc.}$$

If both sides of an equality are equal only for certain values of the unknown quantities, then equality is called an equation. For example $2x + 3 = 15$ is true only for $x = 6$. The letters of alphabet *x, y, z* etc. are generally used for unknown quantities, but *a, b, c* etc. are used for constant numbers or known values. Here $2x + 3 = 15$ is called a linear equation of one unknown. Its solution, i.e. the root of the equation is 6. Similarly $x + 5 = 10$, $3x + 1 = 0$, $\frac{x+3}{2}=4$ are the examples of linear equation. The standard form of linear equation is $ax + b = 0$ (where $a \neq 0$). An equation of the form $ax + by + c = 0$, where *a, b, c* are real numbers, where *a* and *b* are not both zero is called a linear equation of two unknowns *x* and *y*. For an example $2x+3y+5=0$ is a linear equation with respect to the variable *x* and *y*. Similarly $\frac{x}{5}+\frac{y}{7}=1,\ 5x+\frac{y}{2}=6,$ $\frac{x}{2}+15y=9,\ y=4x+7$ etc. are all linear equation of two variables *x* and *y*. We have seen that every linear equation in one unknown or one variable has a unique solution. In case of linear equation involving two variable, a solution means a pair of values, one value for *x* and one value for *y*, which satisfy the given equation. Let us consider the equation $3x+2y=24.$ Here $x = 4$ and $y = 6$ is a solution of the equation. If we put $x = 4$ and $y = 6$ in the left side of the equation we get 24. This solution can be written as an order pair (4, 6), the 1st one value for *x* and 2nd one is for *y*. Similarly (0, 12), (8, 0), (2, 9) etc. are also the solution of the above equation. If we put $x = 1$ in the equation $3x+2y=24,$ then it becomes $3.1 + 2y = 24$, on solving this we get $y=\frac{21}{2}$, therefore $\left(1,\frac{21}{2}\right)$ is a solution of the equation. In this way we can find many solution of the above equation. That is, a linear equation in two variable has infinitely many solution.

Example 1 : Write the equation $2 = 7x + 3y$ in the form of $ax+by+c=0$

Solution: $2 = 7x + 3y$ can be written as

$$7x + 3y - 2 = 0$$

Here $a = 7$, $b = 3$, and $c = -2$

which is in the form of $ax + by + c = 0$.

Example 2 : Find two solution of the equation $3x + 4y = 12$

Solution: $3x + 4y = 12$

Taking $x = 0$, we get $4y = 12$ or $y = 3$.
So (0, 3) is a solution of the given equation.

Similarly, taking $x = 1$, we get $3 + 4y = 12$ or $4y = 9$ or $y = \frac{9}{4}$

So $\left(1, \frac{9}{4}\right)$ is a solution of the given equation.

Exercise 4.1

1. Which of the following is not a linear equation?

(i) $2x = x + a$ (ii) $y^2 + y + 1 = y^2$ (iii) $(x-1)^2 = 4$

(iv) $(x + a)^2 = (a + b)^2$ (iv) $\sqrt{x+4} = x$

2. Write the following equation in the form of $ax + by + c = 0$

(i) $x = 1$ (ii) $y + 4 = 3$ (iii) $\frac{x}{2} + \frac{y}{3} = 7$

3. Find two solution for each of the following :

(i) $x + y = 10$ (ii) $2x + 3y = 5$ (iii) $x + 3y = 5$

(iv) $y = 4x + 3$ (v) $\frac{x}{2} + \frac{y}{3} = 1$ (vi) $y = 3x$

4.2. **Simultaneous linear equation :** A pair of linear equation in two unknowns *x* and *y* form a simultaneous linear equation as follows

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are all real numbers and $a_1^2 + b_1^2 \neq 0,\ a_2^2 + b_2^2 \neq 0.$ There are three algebraic methods for the solution of such equation as

(i) Method of substitution

(ii) Method of elimination

(iii) Method of cross-multiplication

4.2.1 **Method of substitution :** To solve the simultaneous equation of x and y, find the value of one variable, say y in terms of x, from any one equations (whichever is convenient). Now substitute the value of y in the other equation, then the equation become one variable i.e. in terms of x, which can be solved. After getting value of x, substitute the value of x in any one equation, then we get the value of y.

Example 3 : Solve the equations by method of substitution

$$2x+3y=9 \rightarrow (1)$$

$$3x+y=10 \rightarrow (2)$$

Solution: We express y in terms of x, as it is convenient from equation (2)

$$3x+y=10$$

or $y=10-3x \rightarrow (3)$

Now substitute the value of y in equation (1)

we get $2x+3(10-3x)=9$

or $2x+30-9x=9$

or $-7x=9-30$

or $x=\dfrac{-21}{-7}=3$

Now put the value of $x = 3$ in equation (3)

we get $y=10-3.3$

$=10-9=1$

Therefore the solution is $x = 3$; $y = 1$

Application of linear simultaneous equation in two variables.

Example 4 : Two numbers are in the ratio 2 : 5. When 6 is added to both the numbers their ratio becomes 5 : 8. Find the numbers.

Solution: Let the two numbers be x and y

According to the given condition $\dfrac{x}{y}=\dfrac{2}{5}$ or $5x=2y \rightarrow (1)$

Now from 2nd condition of the problem, we get

$$\frac{x+6}{y+6}=\frac{5}{8} \rightarrow (2)$$

or $8(x+6)=5(y+6)$

or $8x+48=5y+30$

or $8x-5y=-18$

Now substitute the value of y from equation (1) we get

$$\because 5x=2y$$

$$y=\frac{5x}{2}$$

$$8x-5\left(\frac{5x}{2}\right)=-18$$

or $\frac{16x-25x}{2}=-18$

or $-9x=-36$

or $x=\frac{-36}{-9}=4$

Therefore $y=\frac{5}{2}\times x=\frac{5}{2}\times 4=10$

The required numbers are 4 and 10.

Exercise 4.2

1. Solve the pair of linear equations by method of substitution.

(i) $x+y=5$
$x-y=1$

(ii) $2x+5y=25$
$x+4y=17$

(iii) $\frac{x}{3}+\frac{y}{5}=\frac{14}{5}$
$2x+3y=3$

(iv) $\frac{x}{3}+\frac{y}{4}=1$
$\frac{x}{4}+y=4$

2. (i) Two numbers are in the ratio 1 : 2, when 5 is added to both the numbers, their ratio becomes 1 : 3. Find the numbers.

(ii) The sum of the two digit number is 11. If the digit are interchange the number is increased by 63. Find the number.

(iii) The present age of a mother is four times that of his son. 5 years ago, the mother's age was seven times the son's age. Find their present age.

4.2.2 **Method of elimination :** To solve the simultaneous equation, first multiply both the equation by some non-zero number to make the coefficient of one variable numerically equal. Now add or subtract one equation from other, so that one variable get eliminated. Then solve the equation of one variable. Now substitute the value in the given equation. Then we get the solution of the equation.

Example 5 : Solve the equations by method of elimination

$$3x+4y=8 \rightarrow (1)$$

$$2x+3y=5 \rightarrow (2)$$

Solution: To solve the simultaneous equation (1) and (2), first we multiply equation (1) by 2 and equation (2) by 3 to make the coefficient of x equal. Then we get the equation as

$$6x+8y=16 \rightarrow (3)$$

$$6x+9y=15 \rightarrow (4)$$

Now subtracting equation (4) from equation (3) we get

$$(6x+8y)-(6x+9y)=16-15$$

or $(6x-6x)+(8y-9y)=1$

or $-y=1$

or $y=-1$

Now, put the value of y in equation (1) we get

$$3x+4(-1)=8$$

or $3x-4=8$

or $3x=8+4=12$

or $x=\frac{12}{3}=4$

$\therefore$ The required solution $x=4,\ y=-1$.

4.2.3 **Method of cross-multiplication :** To solve the pair of equation by method of cross-multiplication, first, we consider the simultaneous equation as

$$a_1x+b_1y+c_1=0 \rightarrow (1)$$

$$a_2x+b_2y+c_2=0 \rightarrow (2)$$

Step 1 : Multiply equation (1) by a_2 and equation (2) by a_1 we get

$$a_2a_1x+a_2b_1y+a_2c_1=0 \rightarrow (3)$$

$$a_1a_2x+a_1b_2y+a_1c_2=0 \rightarrow (4)$$

Step 2 : Subtracting equation (4) from equation (3) we get

$$\left(a_2a_1x-a_1a_2x\right)+\left(a_2b_1y-a_1b_2y\right)+a_2c_1-a_1c_2=0$$

or $\left(a_2b_1-a_1b_2\right)y=a_1c_2-a_2c_1$

or $y=\frac{a_1c_2-a_2c_1}{a_2b_1-a_1b_2}$ Provided $a_2b_1-a_1b_2 \neq 0$

$$=\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}$$

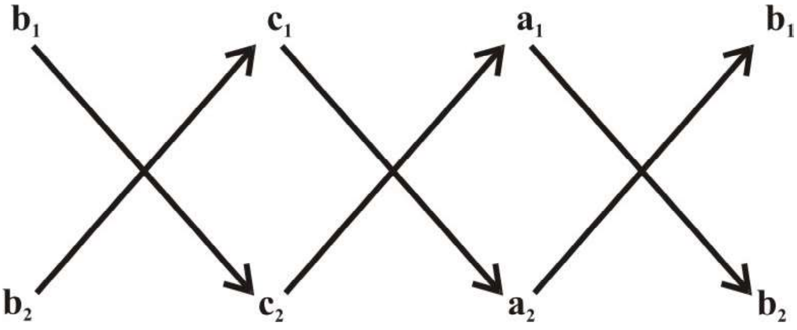
Step 3 : Subtracting this value of y in equation (1) we get

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

Now, combining the value of x and y we can express the relation as

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

The following diagram may be helpful to use the method.



Because of above cross-multiplication, it is called method of cross-multiplication.

Now two cases arise

Case 1 : If $a_1b_2 - a_2b_1 \neq 0$ or $\frac{a_1}{a_2} \neq \frac{b_1}{a_2}$, then the pair of linear equation has unique solution.

Case 2 : If $a_1b_2 - a_2b_1 = 0$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$ then $a_1 = ka_2$, $b_1 = kb_2$. Now, putting the value of a_1 and b_1 in equation (1), we get

$$k(a_2x + b_2y) + c_1 = 0 \rightarrow (5)$$

If $c_1 = kc_2$, then solution of equation (5) as well as equation (2) will satisfy equation (1).

$\therefore$ If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$, then there are infinitely many solution to the pair of equation (1) and (2). But if $c_1 \neq kc_2$ then any solution of equation (1) will not satisfy the equation (2), so the pair has no solution.

Table for nature of solution :

Ratio of coefficient	Nature of equation	Nature of solution
$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Consistent	Unique solution
$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Consistent	Infinite solution
$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Inconsistent	No solution

Example 6 : Solve the equations by method of cross-multiplication

$$3x + 8y = 5$$

$$2x + 4y = 3$$

Solution: The given equation are

$$3x + 8y - 5 = 0$$

$$2x + 4y - 3 = 0$$

Here $\frac{a_1}{a_2} = \frac{3}{2}$, and $\frac{b_1}{b_2} = \frac{8}{4}$, $\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\therefore$ The system has unique solution and is consistent.

Now applying cross-multiplication method for equation (1) and (2) we get

$$\frac{x}{8.(-3) - (-5).4} = \frac{y}{(-5).2 - (-3).3} = \frac{1}{3.4 - 8.2}$$

or $\frac{x}{-24 + 20} = \frac{y}{-10 + 9} = \frac{1}{12 - 16}$

or $\frac{x}{-4} = \frac{y}{-1} = \frac{1}{-4}$

or $x = 1,\ y = \frac{1}{4}$

$\therefore$ The required solution is $\left(1, \frac{1}{4}\right)$

Example 7 : Solve the following simultaneous equations

$$2x + y = 3$$

$$4x + 2y = 6$$

Solution: The given equations are

$$2x + y - 3 = 0 \quad \rightarrow \ (1)$$

$$4x + 2y - 6 = 0 \quad \rightarrow \ (2)$$

Here $\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2}, \ \frac{b_1}{b_2} = \frac{1}{2}, \ \frac{c_1}{c_2} = \frac{-3}{-6} = \frac{1}{2}$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$

$\therefore$ The system of equation has infinite solutions.

Here equation $(2) \div 2 \Rightarrow \frac{4x + 2y - 6}{2} = \frac{0}{2}$

$$\Rightarrow 2x + y - 3 = 0$$

which is same as equation (1)

$\therefore$ Any solution of equation (1) satisfied the equation (2)

$\therefore$ Some solution of the pair of equation is

$\left.\begin{matrix} x = 0 \\ y = 3 \end{matrix}\right\}, \left.\begin{matrix} x = 1 \\ y = 1 \end{matrix}\right\}, \left.\begin{matrix} x = 2 \\ y = 1 \end{matrix}\right\}, \left.\begin{matrix} x = 3 \\ y = -3 \end{matrix}\right\}$ and so on.

Example 8 : If possible solve the following pair of equations

$$2x + y = 3$$

$$4x + 2y = 7$$

Solution: The given equations are

$$2x + y - 3 = 0 \quad \rightarrow \ (1)$$

$$4x + 2y - 7 = 0 \quad \rightarrow \ (2)$$

Here $\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2}, \ \frac{b_1}{b_2} = \frac{1}{2}, \ \frac{c_1}{c_2} = \frac{-3}{-7} = \frac{3}{7}$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2},$

which indicate that, system of equation has no solution.

Example 9 : The sum of the digits of a two digit number is 9. If the digits are interchanged, the number is increased by 27. Find the number.

Solution: Let the unit place of the digit be x and the ten's place be y.

$\therefore$ The number is $10y + x$

According to the first condition

$$x + y = 9 \rightarrow (1)$$

According to the second condition

$$(10x + y) - (10y + x) = 27 \rightarrow (2)$$

or $9x - 9y = 27$

or $9(x - y) = 27$

or $x - y = \frac{27}{9} = 3 \rightarrow (3)$

Now adding equation (1) and (3) we get

$$2x = 12$$

$$x = \frac{12}{2} = 6$$

$$\therefore y = 10 - x = 10 - 6 = 3.$$

$\therefore$ The required number is 36.

Exercise 4.3

1. Solve by elimination method and cross-multiplication method.

(i) $3x + y = 11$, $2x + 3y = 12$ (ii) $5x - 6y = 8$, $4x - 3y = 7$ (iii) $3x + 8y = 1$, $4x + 7y = -6$

(iv) $\frac{x}{4} + \frac{y}{3} = 1$, $\frac{x}{2} + \frac{y}{8} = 2$ (v) $2x + 3y = 8$, $6x + 7y = 18$ (vi) $\frac{4}{x} + \frac{3}{y} = 5$, $\frac{2}{x} + \frac{7}{y} = 9$

2. If possible solve the following simultaneous equations.

(i) $2x + y = 2$, $6x + 3y = 6$ (ii) $3x + 2y = 7$, $9x + 6y = 25$

(iii) $x + 4y = 17$, $2x + 8y = 34$ (iv) $x + 2y = 3$, $2x + 4y = 16$

3. (i) The sum of the digits of a two digit number is 9. If the digits are interchanged, the number increased by \45. Find the number.
   (ii) A man wanted to purchase 6 chair and 6 tables from a market. But for this he was short of Rs. 800. He then decided to purchase 6 chair and 4 tables with Rs. 5800 which he had with him at that time. Find the cost of one chair and one table.
   (iii) The monthly salary of two persons are in the ratio 3 : 5. If each receives an increase of Rs. 200 in the monthly salary the ratio become 13 : 21. Find their salaries.
   (iv) Jadu and Madhu together can do a piece of work in 12 days. They work together for 5 days. When Jadu leaves and Madhu finished the work in 12 days more. In how many days can Jadu alone finish the piece of work?
   (v) A man and a woman can together do a piece of work in 15 days in which 7 men and 9 women can together do in 2 days. In how many days can one man do the work?

Quadratic Equation and its Application

A quadratic equation is of the form $ax^2 + bx + x = 0$ where *a, b, c* are real numbers $(a \neq 0)$ in the variable *x*. For example $2x^2 - 4x + 5 = 0,\ 3x^2 + 8x - 7 = 0,$ and $x^2 - 1 = 0$ etc. are quadratic equation.

$$ax^2 + bx + c = 0\ (a \neq 0) \rightarrow \text{(i)}$$

is called the standard form of quadratic equation. If $b = 0$ in equation (1) is called a pure quadratic. If $b \neq 0$ in equation (1) is called adfected or mixed quadratic equation.

4.3 **Solution of pure quadratic equation :** All pure quadratic equation are reducible to the form $x^2 - d = 0$ (where d = constant).

The equation $x^2 - d = 0$

or $x^2 = d$

or $x = \pm\sqrt{d}$

Example 1 : Solve (i) $x^2 - 2 = 0$ (ii) $2x^2 - 18 = 0$

Solution: (i) $x^2 - 2 = 0$

or $x^2 = 2$

or $x = \pm\sqrt{2}$

(ii) $2x^2 - 18 = 0$

or $2x^2 = 18$

or $x^2 = \frac{18}{2} = 9$

$\therefore x = \pm\sqrt{9} = \pm 3$

4.4 Solution of adfected or mixed quadratic equation :

There are two method for solving mixed quadratic equations.

(a) Solution by factorization method

(b) Solution by completing the square

(a) **Solution by factorization method :** If the left-hand side of the equation $ax^2 + bx + c = 0 \ (a \neq 0)$ can be readily factorized into two linear factors it becomes very easy to solve the equation. We know that if the product of any two expression is zero, atleast one of them must be zero. If P.Q = 0, then either P = 0 or Q = 0, or P and Q both zero.

The method will be illustrated with the help of an example.

Example 2 : Solve $2x^2 - 5x + 2 = 0$

Solution: $2x^2 - 5x + 2 = 0$

or $2x^2 - 4x - x + 2 = 0$

or $2x(x-2) - 1(x-2) = 0$

or $(x-2)(2x-1) = 0$

Now $(x-2)(2x-1) = 0$

This is true if

either $x - 2 = 0$ or $2x - 1 = 0$

From $x - 2 = 0$ or $x = 2$

and from $2x - 1 = 0$ or $2x = 1$ or $x = \frac{1}{2}$

Here $x = 2, \frac{1}{2}$ satisfies the original equation.

Hence the roots are $x = 2, \frac{1}{2}$

(b) **Solution by completing the square :** It is observed that, in certain equations it is not easy to express the expression $ax^2 + bx + c \ (a \neq 0)$ as the product of two linear factors. Such type of equation or say any quadratic equation can be solved by the method of completing square.

Example 3 : Solve by completing the square method $x^2 + 4x + 1 = 0$

Solution: $x^2 + 4x + 1 = 0$

or $x^2 + 4x = -1$

or $x^2 + 2.2.x + 2^2 = -1 + 2^2$

or $(x+2)^2 = 3$

or $x + 2 = \pm\sqrt{3}$

or $x = -2 \pm \sqrt{3}$

$\therefore$ The roots are $-2+\sqrt{3},\ -2-\sqrt{3}$

Let us solve the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ by the method of completing square.

$$ax^2 + bx + c = 0 \quad \rightarrow (1)$$

or $ax^2 + bx = -c$

or $x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad (\because a \neq 0)$

or $x^2 + 2.x.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \quad \left[\text{adding}\left(\frac{b}{2a}\right)^2 \text{ to both side}\right]$

or $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

or $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

or $x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{2a}}$

or $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \rightarrow (2)$

$\therefore$ The roots are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

The relationship shown above in equation (2) is called the formula for solving the quadratic equation or quadratic formula.

Alternatively : Multiply the equation $ax^2 + bx + c = 0$ by $4a$ and transpose $4ac$, then

$$4a^2x^2 + 4abx = -4ac$$

or $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$ [adding b^2 both side]

or $(2ax)^2 + 2.2ax.b + b^2 = b^2 - 4ac$

or $(2ax + b)^2 = b^2 - 4ac$

or $2ax + b = \pm\sqrt{b^2 - 4ac}$

or $2ax = -b \pm \sqrt{b^2 - 4ac}$

or $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The alternative method known as Hindu method or Hindu formula was established by the famous Hindu Mathematician Sridhar Acharyya.

A quadratic equation $ax^2 + bx + c = 0$ has

(i) Two distinct real roots if $b^2 - 4ac > 0$

(ii) Two equal and real roots if $b^2 - 4ac = 0$

(iii) No real roots if $b^2 - 4ac < 0$

Example 4 : Solve the equations by quadratic formula.

(i) $6x^2 - 11x - 10 = 0$ (ii) $x^2 - 5x + 6 = 0$

Solution: (i) $6x^2 - 11x - 10 = 0$

Here $a = 6,\ b = -11,\ c = -10$

$$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-11) \pm \sqrt{(-11)^2 - 4.6.(-10)}}{2.6}$$

$$= \frac{11 \pm \sqrt{121 + 240}}{12}$$

$$= \frac{11 \pm \sqrt{361}}{12}$$

$$= \frac{11 \pm 19}{12}$$

$$= \frac{30}{12} \text{ and } = \frac{-8}{12}$$

$$= \frac{5}{2} \text{ and } = \frac{-2}{3}$$

(ii) $x^2 - 5x + 6 = 0$

Here $a = \text{a}, \ b = -5, \ c = 6$

$$\therefore \ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4.1.6}}{2.1}$$

or $$x = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{5 \pm \sqrt{1}}{2}$$

$$= \frac{5 \pm 1}{2}$$

$$= \frac{6}{2}, \frac{4}{2}$$

$$= 3, 2$$

4.5 **Equation reducible to quadratic form :**

There are various type of equations which are not exactly quadratic in form, which can be reduced to quadratic form by proper substitution and suitable transformation. Methods of reduction to quadratic form and its solution are illustrated by some examples. In certain cases the solution of this type of equation may not satisfy the original equation. In this case the roots which do not satisfy the original equation should be rejected. Solving some equation imaginary roots may be obtained. We will reject such imaginary roots in our study.

Example 5 : Solve $x^4 - 5x^2 + 6 = 0$

Solution: $x^4 - 5x^2 + 6 = 0$

Putting $x^2 = u,$

We find $u^2 - 5u + 6 = 0$

or $u^2 - 3u - 24 + 6 = 0$

or $u(u-3) - 2(u-3) = 0$

or $(u-3)\ (u-2) = 0$

Either $u - 3 = 0$ or $u - 2 = 0$

or $u = 0$ and $u = 2$

when $u = 3$ when $u = 2$

or $x^2 = 3$ or $x^2 = 2$

or $x = \pm\sqrt{3}$ or $x = \pm\sqrt{2}$

$\therefore$ The roots are $\pm\sqrt{3}, \pm\sqrt{2}$

[Note that a fourth degree equation has four roots]

Example 6 : Solve $x + \sqrt{x} = \frac{1}{2}$

Solution: Putting $u = \sqrt{x}$

We have $u^2 + u = \frac{1}{2}$

or $2u^2 + 2u = 1$

or $2u^2 + 2u - 1 = 0$

or $u = \frac{-2 \pm \sqrt{4 - 4.2.(-1)}}{2.2}$

$= \frac{-2 \pm \sqrt{12}}{4}$

$u = \frac{-1 \pm \sqrt{3}}{2}$

$\therefore\ x = u^2 = \frac{2 \pm \sqrt{3}}{2}$

$\therefore$ Roots are $\frac{2+\sqrt{3}}{2}$ and $\frac{2-\sqrt{3}}{2}$

Example 7 : Solve $3^x + \frac{1}{3^x} = \frac{10}{3}$

Solution: Putting $3^x = u$, we get

$$u + \frac{1}{u} = \frac{10}{3}$$

or $3u^2 + 3 = 10u$

or $3u^2 - 10u + 3 = 0$

or $3u^2 - 9u - u + 3 = 0$

or $3u(u-3) - 1(u-3) = 0$

or $(u-3)(3u-1) = 0$

Either $u - 3 = 0$ or $3u - 1 = 0$

or $u = 3$ or $3u = 1$

or $3^x = 3^1$ or $u = \frac{1}{3}$

or $x = 1$ or $3^x = 3^{-1}$

or $x = -1$

$\therefore$ Roots are $x = 1, -1$.

Example 8 : Solve $x^2 + 2x + 9 + \sqrt{x^2 + 2x + 9} = 12$

Solution: Putting $\sqrt{x^2 + 2x + 9} = u$, we get

$$u^2 + u = 12$$

or $u^2 + u - 12 = 0$

or $u^2 - 3u + 4u - 12 = 0$

or $u(u-3) + 4(u-3) = 0$

or $(u-3)(u+4) = 0$

Either $u - 3 = 0$ or $u + 4 = 0$

$u = 3$ or $u = -4$

[$\because$ u is not +ve, we reject $u = -4$]

$\therefore$ $u = 3$

or $\sqrt{x^2 + 2x + 9} = 3$

or $x^2 + 2x + 9 = 9$

or $x^2 + 2x = 0$

or $x(x+2)=0$

either $x = 0$ or $x + 2 = 0$

or $x = -2$

The given equation satisfied by the value $x=0, -2$

$\therefore$ Roots are $x=0, -2$

Example 9 : Solve $2\sqrt{x+5}-\sqrt{2x+8}=2$

Solution: $2\sqrt{x+5}-\sqrt{2x+8}=2$

or $2\sqrt{x+5}=2+\sqrt{2x+8}$

or $4(x+5)=4+(2x+8)+2.2\sqrt{2x+8}$ [Squaring both side]

or $2x+8=4\sqrt{2x+8}$

or $x+4=2\sqrt{2x+8}$

or $x^2+8x+16=4(2x+8)$

or $x^2=16$

or $x=\pm\sqrt{16}=\pm 4$

The given equation satisfied by the value $x=\pm 4$

$\therefore$ Roots of the equation $x=\pm 4$.

Example 10 : Find two consecutive positive integers, sum of whose square is 265.

Solution: Let the integers be x and $x + 1$

According to the condition

$$x^2+(x+1)^2=265$$

or $x^2+x^2+2x+1=265$

or $2x^2+2x-264=0$

or $x^2+x-132=0$

or $(x+12)(x-11)=0$

either $x + 12 = 0$ or $x - 11 = 0$

or $x = -12$ or $x = 11$

$\because$ x is positive integer, therefore $x = 11$.

$\therefore$ The required consecutive positive integers are 11 and –12.

Exercise 4.4

1. Solve the following equation.

(i) $x^2 - 16 = 0$ (ii) $2x^2 - 72 = 0$

(iii) $x^2 + 5x = 0$ (iv) $3x^2 - 15 = 2x^2 + 10$

2. Solve by method of factorization.

(i) $x^2 - 13x + 36 = 0$ (ii) $x^2 - 12x + 32 = 0$

(iii) $6y^2 - 13y + 6 = 0$ (iv) $2x^2 - 7x + 3 = 0$

(v) $x^2 - 2x + 1 = 0$ (vi) $6x^2 - 11y - 10 = 0$

3. Solve by method of completing square.

(i) $x^2 - 2x - 8 = 0$ (ii) $x^2 + 5x + 6 = 0$

(iii) $2x^2 - 5x + 2 = 0$ (iv) $2x^2 + 5x + 1 = 0$

4. Solve by quadratic formula.

(i) $x^2 + 7x + 10 = 0$ (ii) $3x^2 + 5x - 2 = 0$

(iii) $2x^2 - 9x + 4 = 0$ (iv) $x^2 - 2x - 35 = 0$

(v) $x^2 + 3x + 1 = 0$ (vi) $x^2 - 6x + 1 = 0$

5. Solve the equation.

(i) $x^4 - 12x^2 + 32 = 0$ (ii) $9^x - 10.3^x + 9 = 0$

(iii) $2^{x-2} + 2^{3-x} = 3$ (iv) $x + \sqrt{x} = \frac{6}{25}$

(v) $x + \frac{1}{x} = 6$ (vi) $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$

(vii) $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$ (viii) $9\left(x^2 + \frac{1}{x^2}\right) - 24\left(x + \frac{1}{x}\right) - 2 = 0$

(ix) $\sqrt{3x+1} - \sqrt{x-1} = 2$ (x) $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$

6. Application

(i) The sum of the square of two consecutive positive integer is 61. Find the integers.

(ii) The difference between a proper fraction and its reciprocal is $\frac{16}{15}\cdot$ Find the fraction.

(iii) The sum of squares of three consecutive natural numbers is 590. Find the numbers.

(iv) The product of two consecutive numbers is 240. Find the numbers.

4.6 **Simultaneous quadratic equation:**

Simultaneous quadratic equations are two type

Type 1: One equation is quadratic and other one is linear. This type of equation can be solved by method of substitution. The method will be illustrated with the help of an example

Example 1: Solve $x^2+y^2=25;\ x+y=7$

Solution: $x^2+y^2=25 \quad \rightarrow (1)$

$x+y=7 \quad \rightarrow (2)$

From equation (2) $x+y=7$

or $y=7-x$

Now putting this value in equation (1) we get

$x^2+(7-x)^2=25$

or $x^2+49-14x+x^2=25$

or $2x^2-14x+24=0$

or $x^2-7x+12=0$

or $x^2-3x-4x+12=0$

or $x(x-3)-4(x-3)=0$

or $(x-3)(x-4)=0$

either $x-3=0$ or $x-4=0$

or $x=3$ or $x=4$

When $x=3, y=7-3=4$, when $x=4, y=7-4=3$

Required solution is (3, 4), (4, 3).

Example 2: Solve $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2},\ x+y=10$

Solution: $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2} \rightarrow (1)$

$x+y=10 \quad \rightarrow (2)$

From equation (1) we get

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}$$

or $\frac{\sqrt{x}\sqrt{x}+\sqrt{y}\sqrt{y}}{xy}=\frac{5}{2}$

or $\frac{x+y}{\sqrt{xy}}=\frac{5}{2}$

or $\frac{10}{\sqrt{xy}}=\frac{5}{2}$ [$\because$ $x+y=10$ from (2)]

or $\sqrt{xy}=4$

or $xy=16$

or $x(10-x)=16$ $\because x+y=10$, or $y=10-x$

or $10x-x^2=16$

or $x^2-10x+16=0$

or $x^2-8x-2x+16=0$

or $x(x-8)-2(x-8)=0$

or $(x-8)(x-2)=0$

either $x-8=0$ or $x-2=0$

or $x=8$ $x=2$

When $x=8, y=10-8=2$, when $x=2$, $y=10-2=8$

$\therefore$ Required solution is (8, 2) (2, 8)

Type 2 : When both the equations are quadratic

In this case no specific method is applicable. Here we discuss some example.

Example 3: $x+\frac{4}{y}=3$; $y+\frac{5}{x}=7$

Solution: $x+\frac{4}{y}=3 \rightarrow (1)$

$y+\frac{5}{x}=7 \rightarrow (2)$

From equation (1) we get

$xy+4=3y \rightarrow (3)$

From equation (2) we get

$xy+5=7x \rightarrow (4)$

Now subtract equation (3) from equation (4) we get

$$1 = 7x - 3y$$

$$\therefore 7x = 3y + 1$$

or $x = \dfrac{3y+1}{7} \rightarrow (5)$

Now putting the value of x from (5) to equation (4) we get

$$xy + 5 = 7x$$

or $y\left(\dfrac{3y+1}{7}\right) + 5 = 3y + 1$

or $3y^2 + y + 35 = 21y + 7$

or $3y^2 - 20y + 28 = 0$

or $3y^2 - 14y - 6y + 28 = 0$

or $y(3y-14) - 2(3y-14) = 0$

or $(3y-14)\ (y-2) = 0$

either $3y - 14 = 0$ or $y - 2 = 0$

or $3y = 14$ or $y = 2$

or $y = \dfrac{14}{3}$

when $y = \dfrac{14}{3}$, $x = \dfrac{3.\frac{14}{2}+1}{7}$

$= \dfrac{15}{7}$,

when $y = 2$, $x = \dfrac{3.2+1}{7} = 1$

Required solution (1, 2), $\left(\dfrac{15}{7}, \dfrac{14}{3}\right)$

Example 4: $x^2 + y^2 = 40$; $xy = 12$

Solution: $x^2 + y^2 = 40 \rightarrow (1)$

$xy = 12 \rightarrow (2)$

Putting $y = mx$ in the equation we have

$$x^2 + m^2x^2 = 40 \rightarrow (3)$$

$$x.mx = 12 \qquad \rightarrow (4)$$

Dividing equation (3) by (4) we have

$$\frac{x^2 + m^2x^2}{mx^2} = \frac{40}{12}$$

or $\frac{x^2\left(1+m^2\right)}{mx^2} = \frac{10}{3}$

or $3\left(1+m^2\right) = 10m$

or $3m^2 - 10m + 3 = 0$

or $3x^2 - 9m - m + 3 = 0$

or $3m(m-3) - 1(m-3) = 0$

or $(m-3)\,(3m-1) = 0$

either $m - 3 = 0$ or $3m - 1 = 0$

or $m = 3,$ $m = \frac{1}{3}$

Now, putting $m = 3$ in equation (4) we get

$$3x^2 = 12$$

or $x^2 = 4$

or $x = \pm 2$

For $x = 2,$ $y = mx = 6,$

For $x = -2,$ $y = mx = -6$

Again $m = \frac{1}{3},$ in equation (4), we get

$$\frac{1}{3}x^2 = 12$$

or $x^2 = 36$

$x = \pm 6$

For $x = 6,$ $y = mx = \frac{1}{3}.x = 2$

For $x=-6,\quad y=mx=\frac{1}{3}(-6)=-2$

Required solution (2, 6), (6, 2), (–6, –2), (–2, –6)
Miscellaneous examples

Example 5: Solve $y^2=2^x;\ y^x=4$

Solution: $y^2=2^x \quad \rightarrow (1)$

$y^x=4 \quad \rightarrow (2)$

From (1) we get,

$$y=\left(2^x\right)^{\frac{1}{2}}$$

$$\therefore\ y^x=\left(2^{\frac{x}{2}}\right)^x$$

or $4=2^{\frac{x^2}{2}}$

or $2^2=2^{\frac{x^2}{2}}$

or $2=\frac{x^2}{2}$

or $x^2=4$

or $x=\pm 2$

For $x=2,\quad y^2=2^2 \quad$ or $\quad y=\pm 2$

For $x=-2,\quad y^2=2^{-2} \quad y=\pm\frac{1}{2}$

Required solution (2, 2), (2, –2), (–2, ½), $\left(-2,-\frac{1}{2}\right)$

Example 6: Solve $\frac{1}{x^2}+\frac{1}{y^2}=41;\ \frac{1}{x}+\frac{1}{y}=9$

Solution: $\frac{1}{x^2}+\frac{1}{y^2}=41 \rightarrow (1)$

$\frac{1}{x}+\frac{1}{y}=9 \quad \rightarrow (2)$

Putting $\frac{1}{x}=u$ and $\frac{1}{y}=v,$ we get

$$u^2+v^2=41 \rightarrow (3)$$

$$u+v=9 \rightarrow (4)$$

Now from equation (4) we get

$$v=9-u$$

$$\therefore\ u^2+v^2=41$$

or $u^2(9-u)^2=41$

or $u^2+81-18u+u^2=41$

or $2u^2-18u+40=0$

or $u^2-9u+20=0$

or $u^2-5u-4u+20=0$

or $u(u-5)-4(u-5)=0$

or $(u-5)(u-4)=0$

either $u-5=0$ or $u-4=0$

$\therefore u=5$ and $u=4$

For $u=5,$ $v=9-5=4$

For $u=4$ $v=9-4=5$

$\therefore\ u=5,$ $\therefore\ x=\frac{1}{5},$

$v=4$ $\therefore\ y=\frac{1}{4}$

Again $u=4,$ $\therefore\ x=\frac{1}{4}$

$v=5$ $\therefore\ y=\frac{1}{5}$

$\therefore$ Required solution $\left(\frac{1}{5},\frac{1}{4}\right)\left(\frac{1}{4},\frac{1}{5}\right)$

Exercise 4.5

1. Solve the following equations.

(i) $x^2 + y^2 = 50$

$x + y = 8$

(ii) $x^2 + y^2 = 29$

$x - y = 3$

(iii) $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{5}{2}$

$x + y = 10$

(iv) $\frac{x}{2} + \frac{y}{5} = 5$

$\frac{2}{x} + \frac{5}{y} = \frac{5}{6}$

2. Solve

(i) $x + \frac{9}{y} = 7$

$y + \frac{8}{x} = 5$

(ii) $2x^2 - y^2 = 7$

$3x^2 + 2y^2 = 14$

(iii) $2x + 3y = 12xy$

$x + y = 5xy$

(iv) $x^2 + xy = 12$

$xy - 2y^2 = 1$

(v) $x^2 + y^2 = 34;\ xy = 15$

(vi) $x^2 + y^2 + xy = 84$

$x + y + \sqrt{xy} = 14$

3. Solve the following problems.

(i) $y^2 = 2x;\ y^x = 4$

(ii) $3^x = 9^y$

$5^{x+y+1} = 25^{xy}$

(iii) $x^y = y^2\ ;\ y^{2y} = x^4$

(iv) $\frac{1}{x^2} + \frac{1}{y^2} = 13$

$\frac{1}{x} - \frac{1}{y} = 1$

4. (i) Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.

(ii) Two numbers are in the ratio of 3 : 4 and if 10 be subtracted from each of them the remainders are in the ratio 1 : 3. Find the numbers.

(iii) The ratio of the prices of two horses was 16 : 23. Two years later when the prices of the first had risen by 10% and of the second by Rs. 477, the ratio of their prices become 11 : 20. Find the original prices of the two horses.

ANSWERS

Exercise 4.1

1. Linear equation (i), (ii)

2. (i) $x-1=0,\ a=1,\ c=-1$

(ii) $y+1=0,\ a=1,\ c=1$

(iii) $\frac{x}{2}+\frac{y}{3}-7=0 \quad a=\frac{1}{2},\ \ b=\frac{1}{3},\ \ c=-7$

Exercise 4.2

1. (i) $x=3,\ y=2,$ (ii) $x=5,\ y=3$

(iii) $x=13,\ y=-\frac{23}{3}$ (iv) $x=0,\ y=4$

2. (i) $x=-10,\ y=-20$ (ii) $x=2,\ y=9$ the number 29

(iii) Mother's age = 40, son's age = 10

Exercise 4.3

1. (i) $x=3,\ y=2$ (iv) $x=4\ ,\ \ y=0$

(ii) $x=2,\ y=\frac{1}{3}$ (v) $x=-\frac{1}{2},\ \ y=3$

(iii) $x=-5,\ y=2$ (vi) $x=2,\ \ y=1$

2. (i) Infinite solution (ii) No solution

(iii) Infinite solution (iv) No solution.

3. (i) 27 (ii) Chair Rs. 700
Table Rs. 400

(iii) First Rs. 2400
Second Rs. 4000

Exercise 4.4

1. (i) $x=\pm 4$ (ii) $x=\pm 3$

(iii) $x=0,\ -5$ (iv) $x=\pm 5$

2. (i) $4, 9$ (ii) $8, 4$ (iii) $4, 9$

(iv) $\frac{1}{2}, 3$ (v) $+1, +1$ (vi) $\frac{5}{2}, -\frac{2}{3}$

3. (i) $4, -2$ (ii) $-3, -2$ (iii) $2, \frac{1}{2}$

(iv) $\frac{-5 \pm \sqrt{17}}{4}$

4. (i) $-2, -5$ (ii) $-2, \frac{1}{3}$ (iii) $4, \frac{1}{2}$

(iv) $7, -5$ (v) $\frac{-3 \pm \sqrt{5}}{2}$ (vi) $3 \pm 2\sqrt{2}$

5. (i) $\pm 2\sqrt{2},\ \pm 2$ (ii) $2, 0$ (iii) $2, 3$

(iv) $\frac{1}{25}$ (v) $3 \pm 2\sqrt{2}$ (vi) $8, 27$

(vii) $\frac{9}{13}, \frac{4}{13}$ (viii) $3, \frac{1}{3}$ (ix) $1, 5$

(x) $6, -1$

6. (i) $5, 6$ (ii) $\frac{3}{5}$ (iii) $13, 14, 15$

(iv) $15, 16$

Exercise 4.5

1. (i) $(1, 7), (7, 1)$ (ii) $(5, 2)\ (-2, -5)$

(iii) $(4, 1)\ (1, 4)$ (iv) $(4, 15)\ (6, 10)$

2. (i) $(4,3), \left(\frac{14}{5}, \frac{15}{7}\right)$ (ii) $(2, 1)\ (2, -1)\ (-2, 1)\ (-2, -1)$

(iii) $(0, 0), \left(\frac{1}{2}, \frac{1}{3}\right)$

(iv) $\left(4\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right)\left(-4\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right)$, $(3, 1)\ (-3, -1)$

(v) $(5, 3)\ (3, 5)\ (-5, -3)\ (-3, -5)$

(vi) $(8, 2)\ (2, 8)$

3. (i) $(2, 2)$ (ii) $(2, 1), \left(-\frac{1}{2}, -\frac{1}{4}\right)$

(iii) $(2,2), (-2,2)\left(\frac{1}{2}, -2\right)\left(-\frac{1}{2}, -2\right)$ (iv) $\left(\frac{1}{3}, -\frac{1}{2}\right)\left(\frac{1}{2}, -\frac{1}{3}\right)$

4. (i) $10, 6$ (ii) $12; 16$ (iii) Rs. 848, Rs. 1219

Group B : Statistics
Unit I :
Introduction to Statistics

1.1 Meaning of Statistics

"Statistic" as a word is often used in our everyday life. It has been derived from the Latin word 'Status' which means a group of numbers or figures and those represent some interest of our human interest.

"Statistics has two meaning, i.e., the term 'Statistics' has been defined in singular and in plural sense.

According to Oxford dictionary, in plural sense it means a systematic collection of numerical facts and in singular sense, it is defined as the science of collection, presentation, classifying and analysing data using Statistics.

Below are some of the important definitions used in plural sense:

1. "Statistics are the classified facts representing the conditions of the people in a State....specially those facts which can be stated in number or in tables of numbers or in any tabular or classified arrangement." — Webster
2. Statistics are numerical statements of facts in any department of enquiry placed in relation to each other. — Bouley
3. "By Statistics we mean quantitative data affected to a marked extent by multiplicity of causes." — Yule & Kendall
4. "Statistics are measurements, enumerations or estimates of natural phenomenon, usually systematically arranged, analysed and presented as to exhibit important inter-relationships among them." — A.M. Tuttle
5. "Statistics may be defined as the aggregate of facts affected to a marked extent by multiplicity of causes, numerically expressd, enumerated or estimated according to a reasonable standard of accuracy, collected in a systematic manner, for a pre-determined purpose & placed in relation to each other." — Prof. Horace Secrist

Of all the five definitions, Secrist's definition seems to be the most comprehensive and exhaustive.

We give below some of the definitions of Statistics used in singular sense. (STATISTICS AS STATISTICAL METHODS)

1. "Statistics may be called the science of counting". — Bowley A.L.
2. "Statistics is the science and art of handling aggregate of facts- observing, emuneration, recording, classifying and otherwise systematically treating them."
3. "Statistics is a method of decision making in the face of uncertainty on the basis of numerical

data and calculated risks". — Prof. Ya-Lun-Chou

4. "Statistics may be defined as the science of collection, presentation, analysis and interpretation of numerical data." — Croxton & Cowden

Well, it seems pretty much clear that Croxton & Cowden's definition of statistics is the most scientific & authentic one and the first complete definition of statistics along with its four components i.e., collection, presentation, analysis & interpretation of data.

1.2 Importance and scope of Statistics in Business & Economics

The significance of statistics in recent years, is viewed not as a mere device for collecting numerical data but as a means of developing sound techniques for their handling, analysis and drawing valid inferences from them. The development in statistical studies has considerably increased its scope and importance. It is no longer regarded as the science of statecrafts or a by-product of state administration. Nowadays, it is not only confined to the affairs of the state but is intruding constantly into various diversified spheres of life of social sciences, physical and natural sciences.

Now, we are in a position to discuss briefly the significance of Statistics in two important disciplines viz., Business and Economics.

1. **In Business :**

Statistical ideas and knowledge is very helpful to the businessman. Business executives are relying more and more on statistical techniques for studying the needs and the desires of the consumers and for many other purposes. They formulate different plans and policies using statistics, which in turn helps them in forecasting the future trends & tendencies. As an illustration, suppose a businessman wants to manufacture readymade garments. Before starting with the production process he must have an overall idea as to 'how many garments are to be manufactured', 'how much raw material and labour is needed for that' and 'what is the quality, shape, colour, size, etc. of the garments to be manufactured. Thus, the formulation of a production plan in advance is a must which cannot be done without having quantitative facts about the details mentioned above.

Hence, for becoming successful in business, ideas of statistics are essential.

In industrial sector, statistics is very widely used in 'Quality Control', the technique of statisticall quality control is mainly concerned with determining the extent to which quality goals are being met without necessarily checking every item produced and for indicating whether or not the variations which occur are exceeding normal expectations. It also enables us to decide whether to reject or accept a particular product.

2. **In Economics :**

Statistics plays a major role in economics. Statistics for economics concerns itself with the collection, processing and analysis of specific economic data. It helps us understand and analyze economic theories and denote correlations between variables such as demand, supply, price, output etc. Economic theory provides a framework for understanding how markets function and how economic agents make decisions and statistics provide the tools for testing economic theories using empirical data.

Statistical data and techniques of statistical analysis have proved immensely useful in solving a variety of economic problems such as wages, prices, consumptions, production, distribution of income and wealth etc.

Statistical tools like Index Numbers, Time Series Analysis, Demand Analysis are extensively used for efficient planning & economic development of a country.

Statistics along with Economics have given rise to the development of Econometrics, which is now treated as a separate discipline. The term "Econometrics" was coined by the Economist Ragnar Frisch. He along with Jan Tonbergen, who is also an economist is the founder of this subject.

1.3 Types of Data :

Before understanding the concept of data collection, which is an important task of any statistical investigation or enquiry, we will take a look at what do we mean by the term "data" or more precisely "statistical data".

Statistical data : The numerical facts or measurements obtained in the course of an enquiry into a phenomenon, marked by uncertainty, constitute statistical data.

While deciding about the method of data collection to be used for the study, the reader should keep in mind the types of data viz., primary and secondary data.

(1) Primary data

Primary data are those which are collected afresh and for the first time and thus happen to be original in character. We obtain primary data either through observation or through direct communication with respondents in one form or another or through personal interviews.

(2) Secondary data

Secondary data are those which have already been collected by someone else and are obtained from published or unpublished source in any investigation. Usually published data are available in various publications of the central, state or local governments whereas unpublished data sources may be found in diaries, letters, autobiographies and other public/private individuals and organisations.

1.4 Methods of collecting primary and secondary data

The methods of collecting primary and secondary data differ since primary data are to be originally collected, while in case of secondary data, the nature of data collection work is merely that of compilation. We describe the different methods of data collection, with the prons & cons of each method.

(i) Collection of primary data

The methods that are primarily used for collecting primary data are —

(i) Interview method

(ii) Mailed questionnaire method

(iii) Schedule method

(iv) Observation method

(v) Indirect oral interview techniques

Now, we will briefly discuss about each of the method in details.

(I) Interview method

The interview method of collecting data involved an interviewer asking questions generally in a face to face contact to the other person or persons. This sort of interview may be in the form of direct personal investigation or it may be indirect oral investigation. In the case of direct personal investigation the interviewer has to collect the information personally from the sources concerned. In indirect oral investigation, the interviewer has to cross-examine other persons who are supposed to have knowledge about the problem under investigation and the information, obtained is recorded.

Merits

(i) More information can be gained with greater depth.

(ii) Greater flexibility as the questions can be restructured.

(iii) There is choice for interviewer to decide which person(s) will answer the questions. This is not possible in mailed questionnaire method.

(iv) Supplementary information about the respondent's can also be collected.

Demerits

(i) It is a very expensive and time consuming method.

(ii) Introduction of systematic errors remains a huge burden for the interviewer.

(iii) This method is not adopted when the sample is large & widely spread geographically.

(iv) There is often the possibility of introduction of bias made by the interviewer.

(II) Mailed questionnaire method

This method is one of the most popular among all the methods. It is carried out by sending a questionnaire through mail to the persons concerned and are asked to fill the blank spaces provided to them for the questions. It consists of a list of questions numbered in a definite order on a form. After completing the questions, the respondents are requested to send it back within a specified time.

Merits and demerits of mailed questionnaire method

Merits

(i) There is considerable saving in money i.e., cost involved is less.

(ii) There is no involvement of bias of the interviewer since the answers are filled by the respondents themselves.

(iii) Respondents have adequate time to furnish answers in a well prepared mind.

(iv) Respondents, who are not easily approachable, can also be reached conveniently.

Demerits

(i) Such method can be utilised when the respondents are literate/educated.

(ii) Instances of non-response is often a huge problem face by this method.

(iii) Sometimes, it is difficult to accept whether willing respondents are truely the representative.

(III) Schedule method

Collection of data through this method is very much like the questionnaire method, the only difference is that schedules are being filled in by the enumerators who are specially appointed for the purpose. The enumerators along with schedules contact the respondents, record the replies and fill them up in their own language.

Merits and demerits of schedule method

Merits

(i) This method leads to fairly reliable results in extensive enquires.

(ii) This method is applicable when the respondents are uneducated.

(iii) Chances of non-response is less, as the enumerator personally visits/contacts the respondents.

Demerits

(i) This method is very expensive and is usually adopted in investigation conducted by governmental agencies or by some big organisations.

(ii) The outcome of this method relies upon the skills and intelligence of the enumerators selected for the purpose.

(iii) Bias produced by enumerators could impact the result of data.

(IV) Observation method

This method is primarily used in studies relating to behavioural sciences. As the name suggests, such method is usually adopted by researchers who are involve in qualitative research to gather data about people, objects, events, behaviours etc. Researchers watch, listen, takek notes and also record video/audio in their surroundings to get first hand information on the research topic.

Merits & demerits of observation method

Merits

(i) Subjective bias is eliminated, if observation is done accurately.

(ii) Information obtained under this method relates to what is currently happening. It is not complicated by either the past behaviour or future intentions or attitudes.

(iii) It is easy to organize i.e., it happens in a natural environment, so there is no need to organize anything.

Demerits

(i) It is an expensive method.

(ii) Information provided by this method is very limited.

(iii) Sometimes, this method faces obstacles to collect data effectively in situations when some people are rarely accessible to direct observation.

(V) Indirect oral interview method

Under this method, the primary data is collected through which the investigator approaches third parties who are capable of providing the required information. It is used when the area of investigation is large or the respondents feel uncomfortable to furnish information to the investigator due to some reason. In this method, the investigator prepares a list of questions related to the enquiry and then asks questions from different persons & records their answers. The person from whom these questions are asked is known as the witness.

Merits & demerits of indirect oral interview method

Merits

(i) This method is economical in terms of money, time & manpower.

(ii) Prejudices of the original informants are eliminated as the information are recorded from the third parties.

(iii) A wide area can be covered within a given time.

Demerits

(i) Improper choices of the witness by bribery, nepotism or undue requests for which the true information obtained may be twisted by them.

(ii) The informations provided by third parties at times may not be reliable.

1.5 Difference between questionnaire and schedules

There is a slight similarity between both the questionnaire and schedule method and because of this fact people sometimes finds it difficult to distinguish between them. But from technical point of view, there is difference between the two. Here, we will look at some of the notable and important points of difference.

1. The questionnaire is generally sent to the informants to be answered by themselves. On the other hand, a schedule is generally filled out by the investigator or the enumerator, after the responses from the informants are recorded.
2. Collecting data through questionnaire is relatively cheap & economical since, money is involved only in preparing the questionnaire and sending the mail to the resondents. But, to collect data through schedule method, money has to be spent in appointing enumerators & imparting training to them. It is relatively more expensive.
3. Non-response is quite high in case of questionnaire as many respondents do not fill up some of the questions and return it back incompletely. On the other hand, non-response is generally low in case of schedules because of the fact that the answers are being filled by the enumerators themselves.
4. Time involved in collecting responses from all the informants is very high because some of the respondents do not feel it mandatory to fill up on time. But in case of schedules, the information is collected well in time as they are filled in by enumerators.

5. Respondent's literacy is a major concern for questionnaire method, since this method is applicable only for literate people, whereas, even illiterate or uneducated people could participate in answering the questions in case of schedule method.
6. Risk of collecting incomplete and wrong information is relatively more under the questionnaire method, particularly when people are unable to understand questions properly. But in case of schedules, the information collected is generally complete and accurate as enumerators can remove the difficulties, if any, faced by respondents in correctly understanding the questions.

(A) Sources of secondary data

The various sources, the investigator needs to look into, when he utilises secondary data for collecting information can be classified under the following two :

(i) Published sources

(ii) Unpublished sources

(i) Published sources

Usually published are available in (a) various publications of the central, state government's (b) various publications of foreign government's or of international bodies, (c) books, magazines and newspapers (d) research articles, journals (e) local bodies like Municipal Corporation.

(ii) Unpublished sources

They are generally found in letters, diaries and also may be available with scholars and research workers, private individuals and organisations.

1.6 Type of Enquiry

There are two principle types of enquiry, we generally need to adopt in any field of inquiry, they are census and sample survey.

Census refers to the complete enumeration of the items in a population i.e., each & every unit of the population is observed and data are obtained from each of the units.

A sample is a finite sub-set of items in a population. The units comprising the sample should be enable to provide a representative of the population. In sample survey, data is obtained only from the selected items.

(A) Sample Survey Vs Complete Enumeration

Following are the main merits of sample survey over complete enumeration.

1. Sampling usually results in saving time, money and labour.
2. Data can be analysed much faster since only a part of the population needs to be examined.
3. Results obtained from sample survey are much more reliable than those obtained from complete census.
4. When the area of investigation is large, then sampling method is preferred.
5. Sample survey provides more scope as compared with complete census. The complete census is impractible rather inconceivable if the survey requires a highly trained personnel

and more sophisticated equipment for the collection and analysis of the data.

4. If testing is destructive, i.e., if the quality of an article can be determined only by destroying the article in a process of testing, as for example (a) testing the quality of milk (b) testing the breaking strength of chalks (c) testing of crackers, then sampling technique is the only method to be used in such cases.

(B) Limitations of Sampling

Sampling theory has its own limitations and problems which may be briefly outlined are as follows :

1. For the accuracy of results, the investigator should keep in kind that proper care has to be taken in the planning and execution of the sample survey.
2. Sampling method requires the services of trained and qualified personnel and the unavailability of such personnel, the results obtained are not trustworthy.
3. If the information is required about each & every item of the population, there is no way but to resort to complete census.
4. If the universe is not too large, a complete census may be better than sample survey.

Exercise

1. Define statistics.
2. Define Statistics in singular as well as in plural sense.
3. Define Statistics according to Professor Horace Secrist.
4. Mention the scopes & importance of Statistics in Business.
5. Write a brief note on the importance & scope of Statistics in Economics.
6. What are the different types of data used in a statistical investigation?
7. What do you mean by 'statistical data' ?
8. Differentiate between primary and secondary data.
9. Write the methods used for the collection of primary data.
10. Write a brief note on Interview method.
11. Mention the merits and the demerits of Interview method.
12. Write the main differences between Questionnaire and Schedule method.
13. Mention some sources of secondary data.
14. Mention the two principle types of enquiry.
15. Define census.
16. Define sample survey.
17. Write the advantages of sample survey over complete census.
18. Mention the drawbacks of sampling technique.

Unit - II :

Frequency distribution, Diagrams & Graph

Before introducing "what is frequency distribution", we will begin with some ideas about variable and its types.

A variable is a measurable quantity like students marks in a test or height of a girl which varies from one individual to another is called a variable.

(A) Types of variable

(i) Discrete variable: Variables which can assume only a finite no. of distinct integral values, e.g. No. of books in a library.

(ii) Continuous variable: Variables which can assume any value within a specified range, this includes integral as well as fractional values, e.g. temperature of a particular location.

(B) Frequency

The number of times a variable under study appears or is being repeated is called the frequency of the variable.

E.g. (i) If the height of 4 boys lies within the range '160cm - 170 cm' then the frequency of heights in that range is 4.

(ii) If there are 7 students securing 85 marks in Statistic subject, then the frequency of the mark 85 is 7.

2.1 Frequency distribution

Frequency distribution are visual displays that organise and present frequency counts so that the information can be interpreted more easily. It can show absolute frequencies or relative frequencies, such as proportions or percentages.

Types of frequency distribution

There are two types of frequency distributions :

(i) Discrete frequency distribution

(ii) Continuous frequency distribution

(i) Discrete frequency distribution:

When a frequency distribution is constructed for a discrete variable, it is known as discrete frequency distribution.

A discrete frequency distribution may be presented with class intervals or without class intervals.

E.g. (i) Suppose in a class test, marks secured by 30 students is distributed in the following frequency table.

Frequency Table

Marks (x)	No. of students (f)
101 - 15	6
16 - 20	7
21 - 25	4
26 - 30	2
31 - 35	5
36 - 40	2
46 - 50	4
	$\sum f = \mathrm{N} = 30$

Such a table showing the distribution of the frequencies in the different classes is called a frequency table and the manner in which the class frequencies are distributed over the class intervals is called the grouped frequency distribution of the variable.

Suppose, we want to represent the data of the no. of students and their favourite subjects.

Favourite subjects(x)	No. of students (f)
Mathematics	17
Science	13
English	10
Assamese	15
Hindi	12
Social Science	08
	$\sum f = \mathrm{N} = 75$

A frequency distribution of above kind without class intervals is known as an ungrouped frequency distribution.

(ii) Continuous frequency distribution :

When a frequency distribution is constructed for a continuous variable, it is known as continuous frequency distribution. If we are dealing with continuous variable, then it is not possible to arrange the data in the class intervals of the following type.

Age (in years)
0 - 4
5 - 9
10 - 14
15 - 19
20 - 24

Here, the persons with ages between 4 & 5, 9 & 10, 14 & 15, 19 & 20 are not taken into consideration. In such a case, we form the class intervals in the following table.

0 - 5
5 - 10
10 - 15
15 - 20
20 - 25

Here, all the persons with any fraction of age are included in one group or the other.

Suppose we are given the wages (in Rs.) of 43 workers in classes 2,000 - 3,000, 3,000 - 4,000, 4,000 - 5,000, 5,000 - 6,000, 6,000 - 7,000. Then, we represent the data in the following way by grouping the wages into classes.

Wages (in Rs.)	No. of workers
2,000 - 3,000	3
3,000 - 4,000	5
4,000 - 5,000	10
5,000 - 6,000	10
6,000 - 7,000	5
	$\sum f = \mathrm{N} = 43$

Basic Terminologies associated with a grouped frequency distribution

1. Class: A class is a grouping of values of a series, and the groups thus obtained are bounded by limits.
2. Class interval (C.I) : It is the numericall difference between the upper and lower limits of a class.
3. Class limits : It represents the smallest and largest values in a class interval. For e.g. '0 - 10' has two limits, '0' is the lower class limit and '10' is the upper class limits.
4. Class boundaries : These are the values that separate the classes. The following steps are used to calculate the class boundaries in a discontinuous frequency distribution.

Step 1 : Lower class boundary = lower class limit $-\frac{1}{2}$ [lower limit of 2nd class – upper limit of 1st class]

Step 2 : Upper class boundary = upper class limit $+\frac{1}{2}$ [lower limit of 2nd class – upper limit of 1st class]

For e.g. Consider the class interval (C.I)

26 - 30, 31 - 35, 36 - 40, 41 - 45

The upper and lower class boundary for the 1st class i.e., 26 - 30 will be as follows

The lower class boundary $= 26 - \frac{1}{2}[31 - 30] = 26 - \frac{1}{2} = 25.5$

The upper class boundary $= 30 + \frac{1}{2}[31 - 30] = 30 + \frac{1}{2} = 30.5$

Calculating similarly, the required continuous classes are as follows

25.5 - 30.5, 30.5 - 35.5, 35.5 - 40.5, 50.5 - 45.5

5. Inclusive and exclusive class :

Classes of the type in which both the upper and lower limits are included are called 'inclusive classes'.

For e.g. the class '15 - 19', includes all the values from 15 to 19.

Classes of the type in which the upper limits are excludd from the respective classes and are included in the immediate next class are known as 'exclusive classes'.

For e.g. 0 - 5, 5 - 10, 10 - 15, 15 -20. Here the upper limit of each classes are excluded from the respective classes.

6. Open end class :

Classes of the type in which the lower limit of the first class and the upper limit of the last class is not specified, then these are called open end classes.

For e.g Below 5, 5-10, 10-15, 15-20, 20 & above.

7. Width or magnitude of a class :

The difference between the upper and lower limits of a class interval for a continuous or exclusive classes is known as the width or magnitude of the class.

For e.g. 0 - 5, 5 - 10, 10 - 15, 15 - 20, the magnitude of the class is 5 i.e. (5 – 0 = 5).

In case of discontinuous or inclusive classes, at first the class boundaries have to be determined and then the width of the class interval is the difference between the upper and lower class boundaries.

8. Mid value of a class

The average of the lower and upper limit of a class interval is the mid-value of the class.

i.e. Mid value $= \frac{\text{lower limit} + \text{upper limit}}{2}$

9. Class frequency :

The number of times the items corresponding to a class interval repeat in the series.

For e.g. If 15 people aged 5 - 10, then the frequency for the '5 - 10' interval is 15.

10. Frequency density :

It is defined as the ratio of the frequency of a class to the width of that interval i.e.

$$\text{Frequency density} = \frac{\text{Class frequency}}{\text{Width of the class}}$$

11. Relative frequency :

It is defined as the ratio of the frequency of a class to the total frequency

$$\text{i.e. Relative frequency} = \frac{\text{Frequency of the class}}{\text{Total frequency}}$$

12. Percentage frequency :

100 times the relative frequency is called the percentage frequency.

$\therefore$ Percentage frequency = Relative frequency x 100

13. Cumulative frequency (cf)

The cumulative frequency of a class interval is the sum of the frequencies of previous class intervals and the concerned class intervals. The cumulative frequency of the first class is the frequency itself, the cumulative frequency of the second class is the sum of the previous class and the corresponding class itself i.e., sum of the frequency of the 1st & 2nd class. Similarly, cumulative frequency of the last class is the sum of all the classes, and the cumulative frequency thus obtained is the total frequency (N).

The cumulative frequency of the various classes constitute the cumulative frequency distribution.

Cumulative frequency distribution are of two types —

(i) "Less than" and (ii) "More than" types.

(i) Less than type

In less than cumulative frequency distribution, the cumulative frequency corresponds to the upper limit of the class and the cumulative frequencies are calculated in the same way as mentioned above.

(ii) More than type

In more than cumulative frequency distribution, the cumulative frequency corresponds to the lower limit of the class. The more than cumulative frequencies are obtained by summing up the frequencies from the highest class interval / last class interval and then proceeding upwards.

The cumulative frequencies of the last class is the frequency itself. The cumulative frequencies of the class succeeded by the last class is sum of the frequencies of the last class and the corresponding class. Similarly, the cumulative frequency of the 1st class is the sum of all the frequencies (N).

E.g. Suppose, we have a frequency distribution which is given below. Obtain 'less than cf' & 'more than cf.

Class Intervals (C.I)	(f) frequency	Less than cf	More than cf
0-5	3	3	25 (7 + 9 + 5+ 4 + 3)
5-10	6	9 (6 + 3)	22 (7 + 4 + 5+ 6)
10-15	5	14 (5 + 6 + 3)	16 (7 + 4 + 5)
15-20	4	18 (4 + 5 + 6 + 3)	11 (7 + 4)
20-25	7	25 (7 + 4 + 5+ 6 + 3)	7
	$\sum f = \text{N} = 25$		

Construction of Frequency Tables

For the construction of frequency distribution table, three columns are prepared. The first column consists of the no. of variables or class intervals, the second column consists of the tally marks recorded against each of the variables according to their no. of occurences and the third column consists of the frequencies corresponding to the variables or classes. The values in the first column are arranged from lowest to highest either as single observations or in classes.

Remarks :

1. A bar (1) called tally marks is put against the number when it occurs. If it occurs five times, then put a cross tally on the first four tallies (𝍸)
2. Points to remember for classification
1. The classes should be clearly defined and should not lead to any ambiguity.
2. The classes should be mutually exclusive and non-overlapping.
3. The classes should be exhaustive i.e., each of the given values should be included in one of the classes.
4. The classes should be of equal width.
5. Open end classes like less than '*a*' or greater than '*b*' should be avoided as far as possible since they create difficulty in analysis and interpretation.
6. Number of classes should neither be too large nor too small. It should preferably lie between 5 and 15. However, the number of classes may be more than 15 depending upon the total frequency and the details required.

Remark:

1. Struge's Formula for determining an approximate number '*x*' of classes
   K = 1 + 3.322 $\log_{10}$N, where N = total frequency.

Some worked out problems

Prepare a discrete or ungrouped frequency distribution table for the following marks obtained by 24 students.

19	22	12	30	30	25	24	18
21	30	30	19	18	33	20	19
19	12	21	27	22	29	18	21

Soln: The above marks can be put in a frequency table in the following way

Variable	Tally marks	Frequency
12	II	2
18	III	3
19	IIII	4
20	I	1
21	III	3
22	II	2
24	II	2
25	I	1
27	I	1
30	IIII	4
33	I	1
Total		N = 24

2. Prepare a grouped frequency distribution table from the following marks obtained by 30 students of a class.

46, 67, 23, 05, 12, 53, 38, 59, 26, 43
45, 66, 74, 16, 86, 56, 31, 58, 90, 32
74, 48, 64, 58, 50, 46, 53, 64, 57, 65

Find (i) Percentage of students whose marks is below 50.
(ii) Percentage of students whose marks is above 50.
(iii) Number of students whose marks is between 50 - 59.

Soln: Here, the lowest mark is 05 and the highest mark is 90. The range of the data is 90 - 5 = 85 and hence we can take the length of the class interval as 10.

Frequency table of marks of 30 students

Class Intervals (C.I)	Tally marks	Frequency (f)	Cumulative frequency (cf)
0 - 9	I	1	1
10 - 19	II	2	3
20 - 29	II	2	5
30 - 39	III	3	8
40 - 49	~~IIII~~	5	13
50 - 59	~~IIII~~ III	8	21
60 - 69	~~IIII~~	5	26
70 - 79	II	2	28
80 - 89	I	1	29
90 - 100	I	1	30
Total		N = 30	

(i) No. of students whose mark below 50 is 13

i.e. 1 + 2 + 2 + 3 + 5 = 13 (cf of (40 - 49)

$\therefore$ Percentage of students securing marks below 50 is

$$\frac{13}{30} \times 100 = 43.33\%$$

(ii) No. of students whose mark above 50 is 17

i.e. 8 + 5 + 2 + 1 + 1 = 17

(Total no. of students – No. of students securing less than 50)

i.e., (30 – 13) = 17

$\therefore$ Percentage of student securing marks above 50 is

$$\frac{17}{30} \times 100 = 56.66\%$$

(iii) No. of students securing between 50 - 59 is 8

3. The following numbers give the weights of 55 students of a class. Prepare suitable frequency table.

42	74	40	60	82	115	41	61	75	83	63
53	110	76	84	50	67	65	78	77	56	95
68	69	104	80	79	79	54	73	59	81	100
66	49	77	90	89	76	42	64	69	70	80
72	50	79	52	103	96	51	86	78	94	71

Find also the frequency density and percentage frequency of the class intervals.

Soln: Here, total frequency (N) = 55

By Sturge's rule, the no. of classes (K) is given by

$$K = 1 + 3.322 \log_{10} 55$$

$$= 1 + (3.322 \times 1.7403)$$

$$= 1 + 5.7814$$

$$= 6.7814 \approx 7$$

Range = Maximum value – Minimum value

$= 115 - 40$

$= 75$

The class interval of the class is given by

$$\text{C.I} = \frac{\text{Range}}{\text{K}} = \frac{75}{7} = 10.714 \approx 11$$

Hence, taking the magnitude of each class interval as 11, we shall get 7 classes. Since, the minimum weight is 40, which is a convenient figure to take as lower limit of a class, the various classes by 'exclusive method' would be : 40-51, 51-62, 62-73, 73-84, 84-95, 95-106, 106-117.

Now, we prepare the continuous frequency distribution table of weight of 55 students.

Class Intervals (C.I)	Tally marks	Frequency (f)	Frequency density	Percentage of frequency
40 - 51	𝍸 II	7	$\frac{7}{11} = 0.63$	$\frac{7}{55} \times 100 = 12.72\%$
51 - 62	𝍸 III	8	$\frac{8}{11} = 0.72$	$\frac{8}{55} \times 100 = 14.54\%$
62 - 73	𝍸 𝍸 I	11	$\frac{11}{11} = 1.00$	$\frac{11}{55} \times 100 = 20\%$
73 - 84	𝍸 𝍸 𝍸 II	17	$\frac{17}{11} = 1.54$	$\frac{17}{55} \times 100 = 30.90\%$
84 - 95	𝍸	5	$\frac{5}{11} = 0.45$	$\frac{5}{55} \times 100 = 9.09\%$
95 - 106	𝍸	5	$\frac{5}{11} = 0.45$	$\frac{5}{55} \times 100 = 9.09\%$
106 - 117	II	2	$\frac{2}{11} = 0.18$	$\frac{2}{55} \times 100 = 3.63\%$
Total		55		

Exercise

1. Define variable.
2. What are the types of variable?
3. Define frequency.
4. What do you mean by a 'frequency distribution' ? And what are its different types?
5. Define discrete frequency distribution with a suitable example.
6. Define continuous frequency distribution with a suitable example.
7. State one difference between ungrouped and grouped frequency distribution.
8. Define the following :
   (i) Class
   (ii) Class limits

- (iii) Class boundaries
- (iv) Open end classes

9. Define 'inclusive' and 'exclusive' classes.

10. Define the following :
    - (i) Class frequency
    - (ii) Frequency density
    - (iii) Percentage density
    - (iv) Cumulative frequency

11. What do you mean by 'cumulative frequency' ?

12. Explain how will you construct a frequency distribution table.

13. Prepare the important points for classification.

14. Prepare a frequency distribution table from the following data. Find also the frequency density and percentage frequency of each class interval.

21 20 55 39 48 46 36 54 42 30
29 42 32 40 34 31 35 37 52 44
39 45 37 33 51 53 52 46 43 47
41 26 52 48 25 34 37 33 36 27
54 36 41 33 23 39 28 44 45 39

15. Age at death of 50 persons of a village are given below. Prepare a frequency distribution table showing tally marks, frequency and percentage frequency. Prepare exactly 10 class intervals as 25 - 29, 30 - 34, ..., etc.

30, 71, 40, 58, 48, 47, 43, 55, 63, 57, 49, 54, 34,
70, 63, 68, 62, 28, 38, 48, 44, 39, 52, 51, 36, 37,
57, 49, 60, 51, 29, 37, 61, 37, 51, 60, 46, 44,
57, 55, 39, 56, 49, 68, 65, 32, 33, 53, 45

16. Form a frequency distribution table from the following data & answer the question given below.

Monthly salary of 30 workers in Rupees

310, 320, 325, 354, 370, 335, 300, 397, 331, 375,
315, 390, 350, 386, 359, 360, 380, 323, 342, 327,
305, 318, 337, 376, 392, 340, 363, 385, 369, 393

Find—

- (i) Percentage of workers whose monthly salary is below Rs. 350.
- (ii) Percentage of workers whose monthly salary is above Rs. 350.
- (iii) Number of workers in the salary range 325 - 365.

17. From the following table showing the wage distribution in a certain factory calculate 'less than cf' & 'more than cf' & find

    (i) Percentage of workers who earned less than Rs. 100 per week

    (ii) Percentage of workers who earned more than Rs. 150 per week

    (iii) Percentage of workers who earned between Rs. 75 and Rs. 125.

Weekly wages (Rs.)	No. of employees
20 - 40	8
40 - 60	12
60 - 80	20
80 - 100	30
100 - 120	40
120 - 140	35
140 - 160	18
160 - 180	7
180 - 200	5

Diagrams and Graph

Introduction :

Diagrams and graphs give visual representation of the data. If properly created they can attract and hold the attention of the viewers, simplify the complexity of data. They can give a clear picture of data and can be used to make comparisons easily.

(A) Diagrammatic Presentation of data:

When data is presented in a simple and attractive manner in the form of diagrams is called diagrammatic presentation of data. The form of diagrams is called diagrammatic presentation of data.

2.2 Types of diagram

The commonly used diagrams to present statistical data are classified as follows:

1. **One-dimensional diagrams :**

   This diagrams have only one dimension such as height or length. They consists mainly of bar diagrams. The magnitude of the characteristics is shown by length or height of the bar.
2. **Two-dimensional diagrams:**

   This diagrams have two dimensions such as length, breadth. It consists of rectangles, squares, circles or pie-diagram.
3. **Three-dimensional diagrams :**

   This type of diagrams have three dimensions such as length, breadth & height. It consists of cubes, cylinders, spheres etc.
4. **Pictograms :**

   It represents the frequency of data while using symbols or images that are relevant to the data. Pictograms is one of the simplest ways to represent statistical data.
5. **Cartograms :**

   It refers to a map through which information are represented in different manner viz., shades, dots, columns. The map is usually distorted.

2.3 Bar Diagram

Bar diagrams are one of the most popular one-dimensional diagrams. Businessman and Economist utilized bar diagrams for presenting business and economic data. Here, only the length (or height) of the bars are taken into account.

Following are the points that should be kept in mind while constructing bar diagrams :

1. All the bars should be on the same base line.
2. The length or height of the bars vary depending on the value of the variable. However, the breadth (or width) of the bar remains constant.
3. All the bars must be equally spaced from each other.
4. Bars may be drawn either horizontally or vertically. But in practice, vertical bars are usually adopted.

5. The bars should be arranged from left to right (from top to bottom in case of horizontal bars).

Types of Bar diagrams:

1. Simple Bar diagram
2. Multiple or compound bar diagram
3. Sub-divided or component bar diagram

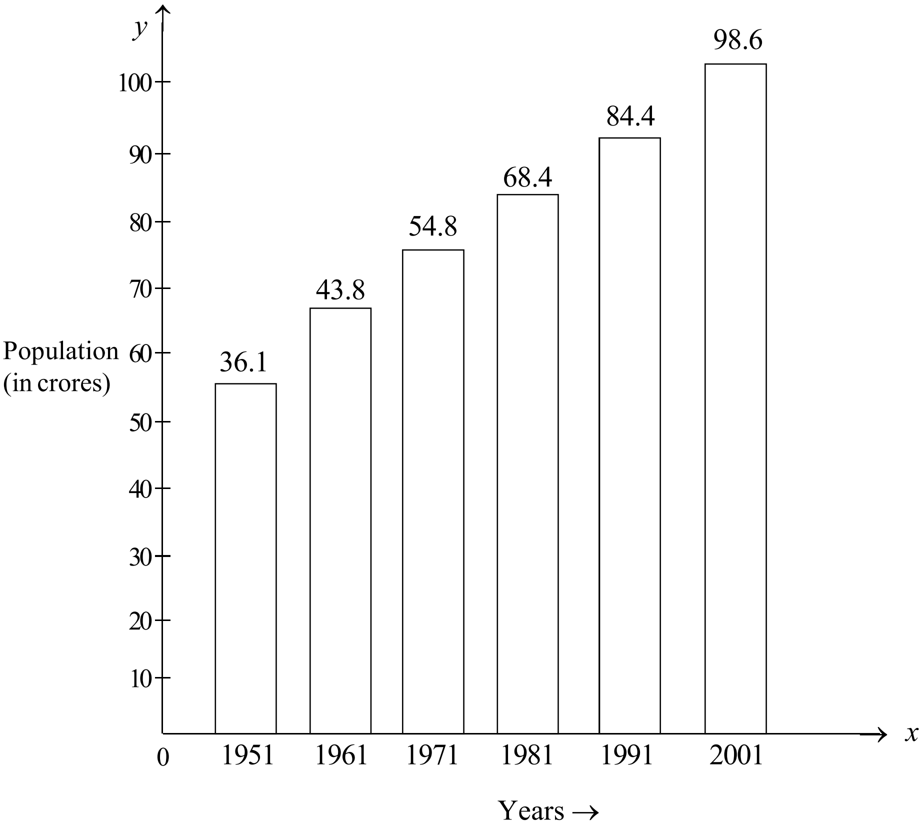
1. Simple Bar diagram

A diagram in which each class or category of data is represented by a group of rectangular bars of equal width is known as a simple bar diagram. In this diagram, each bar represents only one figure. These diagrams show only one characteristics of the data such as sales, production or population figures for various years. The magnitude of data is determined by the bars height (or length). The layout of these diagrams can be vertical or horizontal.

Example 1: Draw a bar diagram for the population of India in different years.

Year	: 1951	1961	1971	1981	1991	2001
Population (in crores)	: 36.1	43.8	54.8	68.4	84.4	98.6

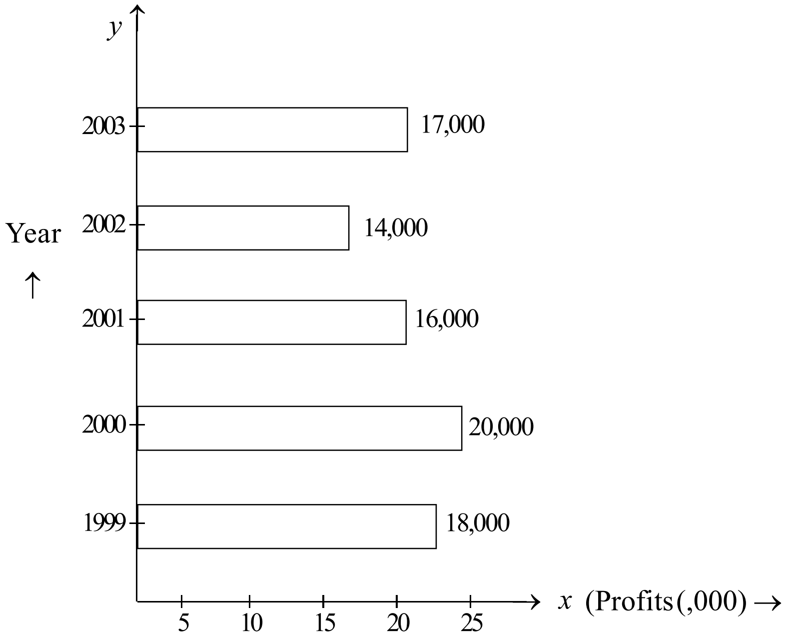
Soln: The bar diagram is drawn as shown below :



Example 2: Draw a horizontal bar to represent the following :

Year	:	1999	2000	2001	2002	2003
Profit (in Rs.)	:	18,000	20,000	16,000	14,000	17,000

Soln: The horizontal bar is drawn as shown below :



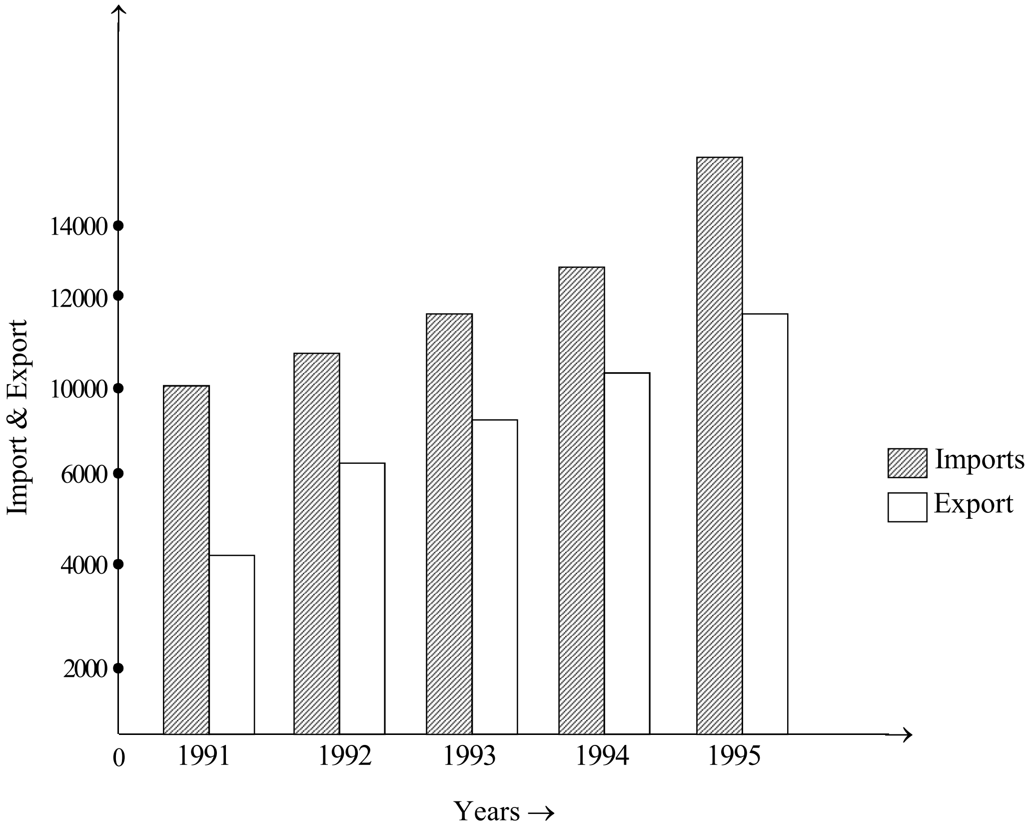
2. **Multiple or compound bar diagram**

The multiple bar diagram is used to compare two or more inter-related variables such as revenue and expenditure, import and export for different years, marks obtained in different subjects and so on. The construction of multiple bar diagram is same as for creating a simple bar diagram. However, to distinguish the bars from each other, different bars are differentiated by different shades or colors.

Example 3: Represent the following data by a multiple bar diagram

Year	Imports	Exports
1991	7930	4260
1992	8850	5225
1993	9780	6150
1994	11720	7340
1995	12150	8145

Soln: The multiple bar diagram for the above data is drawn as below.



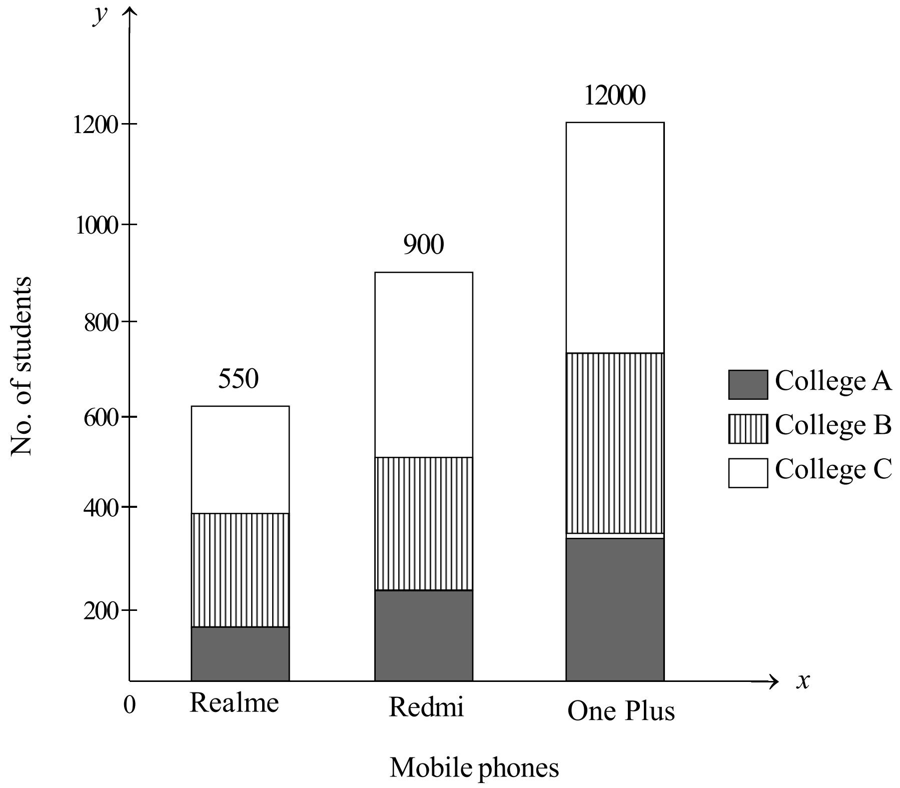
3. Sub-divided or component bar diagram

In these diagrams, the total count or magnitude of the variable under study is divided into several components or parts. Every component occupies a fraction or proportion of the bar to its share in the total. When preparing a sub-divided bar diagram, the various components in each bar should be kept in the same sequence. Also, to distinguish between different components different shades or colors are used.

Example 4: Represent the following information using a sub-divided bar diagram.

Mobile phones	College A	College B	College C	Total
Realme	100	200	250	550
Redmi	200	300	400	900
One Plus	300	400	500	1200

Soln:



2.4 **Pie Diagram**

A pie diagram is a circle which is divided into parts to show the ratios or percentages of different components.. They are also known as Angular Circle diagrams. The circle is divided into as many sections as there are components by drawing straight lines from the centre to the circumference of the circle.

For drawing a pie-diagram, the following procedure is followed:

(i) The 1st step is to calculate the percentage of every value of the total.

(ii) A circle subtends 360° at the centre. Therefore, the total angle i.e., 360° represents 100%.

$$\text{Thus, } 1\% = \frac{360^\circ}{100} = 3.6^\circ$$

Since, 1% is equal to 3.6°, the percentage of every part is multiplied by 3.6° to give the proportionate angles.

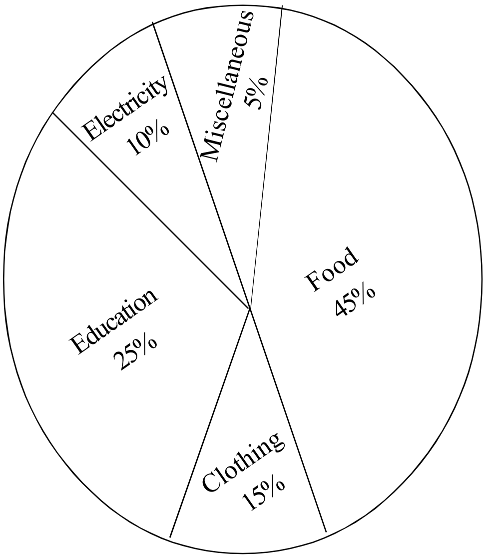
Example 5: Draw a pie chart to reflect the following expenditure for an ordinary working class family.

Items	% of Total Expenditure
Food	45
Clothing	15
Education	25
Electricity	10
Miscellaneous	5

Soln:

Items	% of Total Expenditure	Proportionate Angles
Food	45	$45 \times 3.6 = 162^\circ$
Clothing	15	$15 \times 3.6 = 54^\circ$
Education	25	$25 \times 3.6 = 90^\circ$
Electricity	10	$10 \times 3.6 = 36^\circ$
Miscellaneous	5	$5 \times 3.6 = 18^\circ$
	Total	360°

According to the degrees of angles at the centre, the circle is divided into five parts. Hence, the pie diagram for the given data will be represented as :



(A) Graphical Representation of Data

It is a form of virtually displaying data through the use of graph paper. Graph helps in visualizing, presenting data and to study the existing relationship between two variables through different types of graphs. As compared to diagrams, graphs gives more accurate information. There are different types of graphical representation. Some of the important ones are as follows:

1. Histogram
2. Frequency Polygon
3. Frequency Curve
4. Cumulative frequency curves or ogives.

1. **Histogram :** One of the important and popular methods of summarizing, presenting discrete or continuous data. It shows the frequency of numerical data using rectangles.

(i) Locate the class boundaries on the x-axis. These are taken as the bases.

(ii) Draw a vertical rectangle on each base whose height is proportional to the frequency of the class. These frequencies are marked along the y-axis.

Histogram with equal class intervals:

When the class intervals are of equal length, the drawn histograms are known as Histograms of equal class intervals. Histogram of equal class intervals includes rectangles with equal width; however, the length of the rectangles is proportional to the frequency distribution of the class intervals.

Histogram with unequal class-intervals:

When the class intervals are of unequal length, the drawn histogram are known as Histograms of unequal class intervals. Histogram of unequal class intervals includes rectangles with different width; therefore, before constructing a histogram in case of unequal class intervals, frequency distribution has to be adjusted.

Adjustment of frequencies of unequal class intervals:

1. Determine the class of the smallest interval (lowest class interval)
2. Calculate the adjustment factor using the formula:

$$\text{Adjustment factor for any class} = \frac{\text{Class intervalsof the concerned class}}{\text{Lowestclass interval}}$$

3. Now, adjust the given frequencies using the adjustment factor :

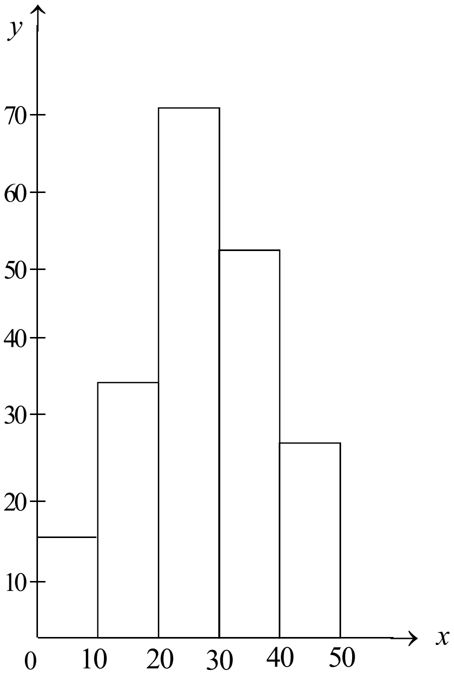
$$\text{Frequency density} = \frac{\text{Given frequency}}{\text{Adjustment factor}}$$

Example 6: Represent the following information in the form of a Histogram.

Marks :	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Number of students	16	36	70	50	28

Soln: To represent the above data by a histogram, we proceed as follows :

The class intervals are of equal size, so plot the class-interval along with x-axis and the frequency along the y-axis.



Note: Appropriate scale must be taken along x-axis & y-axis to draw the histogram in the graph paper. The scale which is taken to draw the histogram must be specified.

Example 7: Represent the following data by a histogram :

Wages	:	10 - 15	15 - 20	20 - 25	25 - 30	30 - 40	40 - 60	60 - 80
No. of workers	:	14	20	54	30	24	24	16

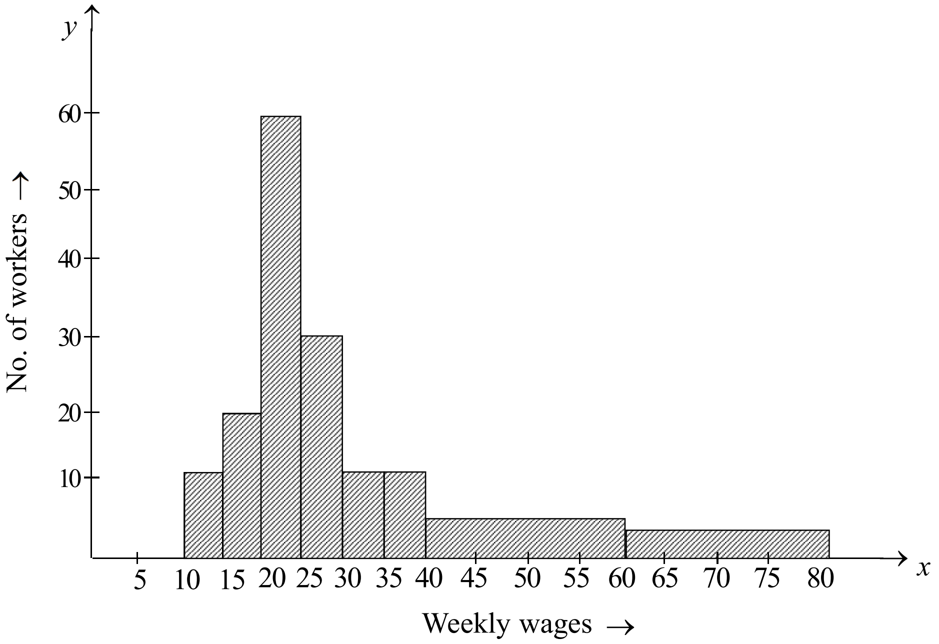
Soln: Clearly, the class-intervals are not of equal magnitude. So, before constructing the histogram, frequencies have to be adjusted.

Lowest class intervals in the given frequency distribution is 5.

Form the adjusted table as shown below :

Wages	Number of workers	Adjustment factor	Frequency density
10 - 15	14	$5 \div 5 = 1$	$14 \div 1 = 14$
15 - 20	20	$5 \div 5 = 1$	$20 \div 1 = 20$
20 - 25	54	$5 \div 5 = 1$	$54 \div 1 = 54$
25- 30	30	$5 \div 5 = 1$	$30 \div 1 = 30$
30 - 40	24	$10 \div 5 = 2$	$24 \div 2 = 12$
40 - 60	24	$20 \div 5 = 4$	$24 \div 4 = 6$
60 - 80	16	$20 \div 5 = 4$	$16 \div 4 = 4$

Now, taking the class-intervals along the x-axis and the frequency densities along the y-axis, a histogram is to be drawn.



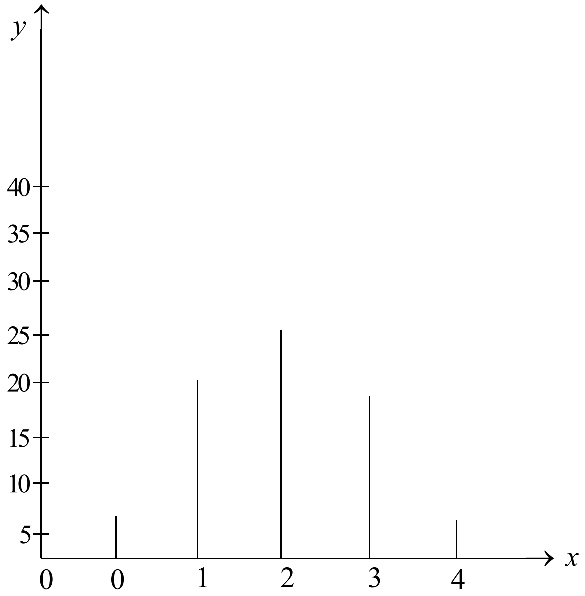
Remarks : (1) If the frequency distribution is discontinuous, convert it into a continuous distribution and then the histogram is drawn on the continuous base.

(2) For an ungrouped discrete distribution, the graphical representation is done in the form of a Rod graph which is illustrted below.

Example 8 : Represent the following data graphically :

No. of heads	Frequency
0	6
1	28
2	36
3	25
4	5
Total	100

Soln: Plotting no. of heads along x-axis and the corresponding frequencies along the y-axis then we have



Frequency Polygon :

Another graphical tool equivalent to a histogram that is used to represent frequency distribution (continuous or discrete). For an ungrouped frequency distribution, the variable values are marked on the x-axis and the frequencies are marked along the y-axis. The points thus obtained are joined by means of a straight lines.

For ungrouped frequency distribution, the mid values of the class intervals are plotted on the x-axis and the corresponding frequencies are represented along the y-axis.

(A) Construction of frequency polygon for grouped frequency distribution following steps are required to form a frequency polygon.

Step 1 : Choose the class intervals and then indicate the values on the ones.

Step 2 : Label the x-axis with the mid-value of each class interval

Step 3 : Label the y-axis with the class frequency

Step 4 : Mark a point at the height in the centre of each class interval according to the frequency of each class interval

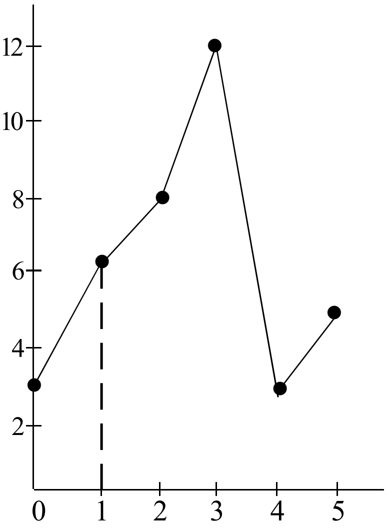
Step 5 : Use the line segment to join these spots

Step 6 : The closed figure so obtained is a frequency polygon.

Example 9 : Represent the following data by a frequency polygon.

Goals scored	:	0	1	2	3	4	5
Frequency	:	3	7	8	12	2	5

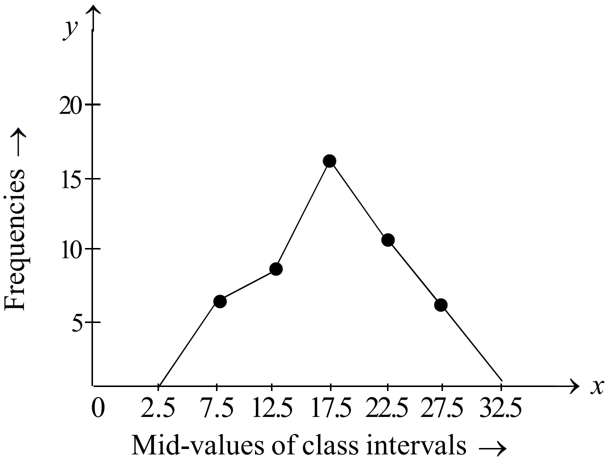
Soln: For the above data, we first plot the frequency polygon by representing the goals scored on the x-axis and frequency on the y-axis as follows.



Frequency Curve :

A smooth, free hand curve drawn through the vertices instead of joining the points by straight lines as (in the case of frequency polygon). In other words, a frequency curve is a limiting form of a histogram or a frequency polygon. A frequency curve has the advantage of showing the skewness of the distribution i.e., whether the curve so obtained is concentrating more towards one side (left or right) than the other.

From example 10, we can construct a frequency curve as given below.



4. Cumulative frequency curves or Ogives:

A cumulative frequency curve is obtained by plotting the different class intervals along the x-axis and the corresponding frequencies along the y-axis and joining these points successively by free hand curve (smooth curve).

There are two types of cumulative frequency curve :

(i) Less than cumulative frequency curve

(ii) More than cumulative frequency curve.

(i) Less than cumulative frequency curve :

In less than cumulative frequency curve, the less than cumulative frequencies are plotted against the upper class limits of the respective classes, i.e., plot the upper class limits of each classes along the x-axis and the corresponding cumulative frequencies along the y-axis, the required curve is obtained by joining these points successively by a smooth curve.

(ii) More than cumulative frequency curve :

In more than cumulative frequency curve, plot the lower class limits of each classes along the x-axis and the corresponding cumulative frequencies along the y-axis, the required curve is obtained by joining these points successively by a smooth curve.

(A) Significance of cumulative frequency curve (ogives)

(i) These curves can be serve as a useful tool for determining partition values (values which the series into a number of equal parts) such as quartiles, deciles and percentiles.

(ii) They can also be used to determine graphically the number of observations that lie above (or below) a particular value in a given data set.

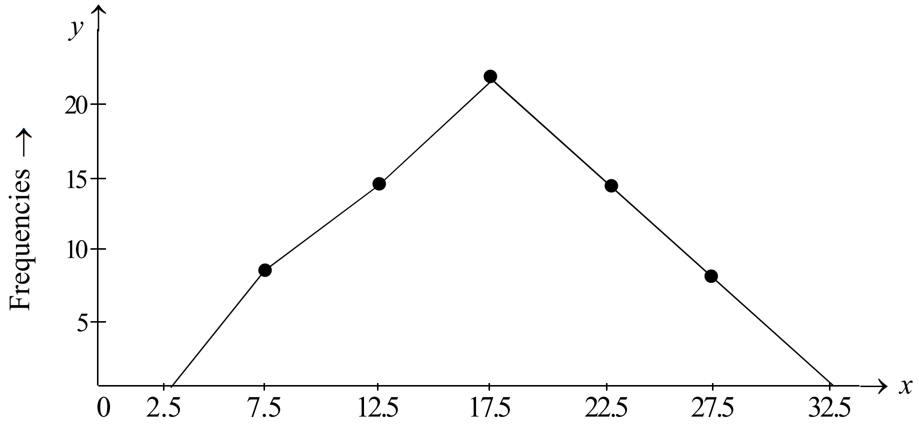
Example 10 : Represent the following data by a frequency polygon.

Class intervals	:	5 - 10	10 - 15	15 - 20	20 - 25	25 - 30
Frequency	:	7	10	18	13	5

Soln: Let us first set up the table by including two more classes and then finding the mid values of all the classes.

Class intervals (C.I)	Mid value (x)	Frequency (f)
0 - 5	2.5	0
5 - 10	7.5	7
10 - 15	12.5	10
15 - 20	17.5	18
20 - 25	22.5	13
25 - 30	27.5	5
30 -35	32.5	0

Now, plotting the mid values along the x-axis and the frequencies along the y-axis, we draw a frequency polygon by taking appropriate scale as follows :



Mid-values of class intervals →

Remark : From the above example, addition of two classes, the lowermost and the uppermost classes are added to the given distribution and these two classes are assumed to have zero frequency. The end points of the line graph already drawn are joined to the middle points of the two added classes to obtain a closed figure which is called a frequency polygon.

Example 11 : Draw the cumulative frequency curve for the following data set.

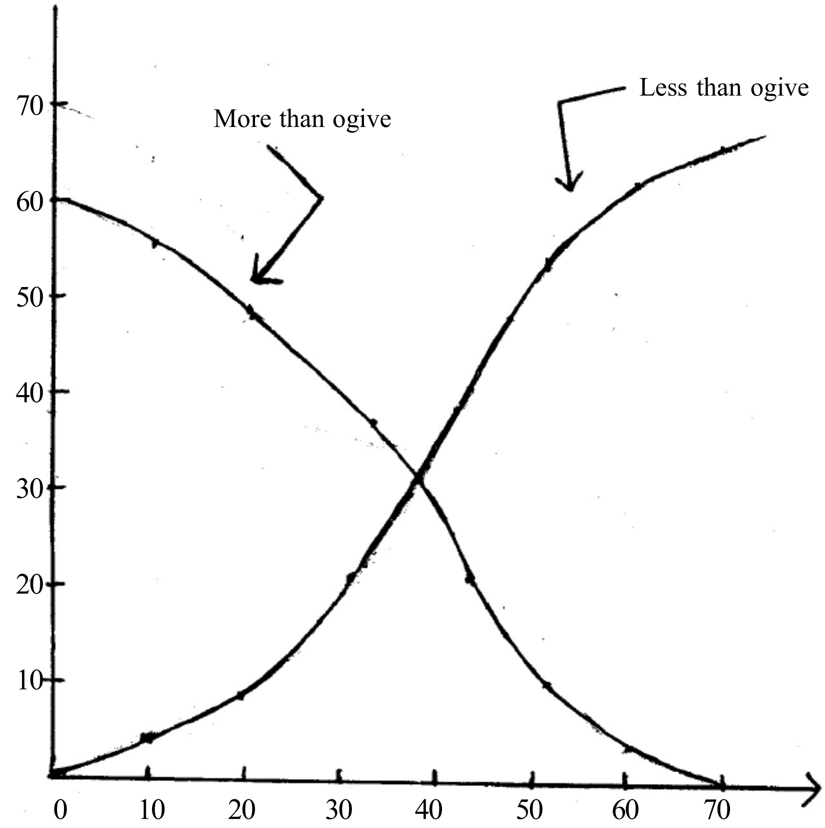
Marks group :	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
No. of students :	4	8	11	15	12	6	3

Soln: The table of cumulative frequencies is as follows :

Marks group	No. of students	Less than *cf*	More than *cf*
0 - 10	4	4	59
10 - 20	8	12	55
20 - 30	11	23	47
30 - 40	15	38	36
40 - 50	12	50	21
50 - 60	6	56	9
60 -70	3	59	3

For obtaining less than cumulative frequency curve, we take the marks along x-axis and cf along y-axis, we plot the less than cumulative frequencies, viz., 4, 12, 23, 38, ..., 59 against the upper limits of the corresponding classes, viz., 10, 20, 30, ..., 70 respectively and join the points by a smooth curve.

For obtaining more than cumulative frequency curve, we take the marks along x-axis and cf along y-axis, we plot the more than cumulative frequencies, viz., 59, 55, 47, ..., 3 against the lower limits of the corresponding classes, viz., 0, 10, 20, 30, ..., 60 respectively and join the points bv a smooth curve.



From the above figure, the point of intersection of both the curve gives the median of the data set.

Exercise

1. What are the purposes does diagrams and graphs serve?
2. Point out some differences between diagrams and graphs.
3. Mention different types of diagrams.
4. What do you mean by bar diagram? Write down the important steps one should keep in mind while constructing a bar diagram.
5. Describe multiple or compound bar diagram.
6. Explain pie chart / diagram. Mention the usual steps in constructing a pie chart.
7. How do you represent data in a graphical form?
8. What are the different types of graphical representation of data?

9. Define Histogram. Differentiate between histogram with equal class intervals and unequal class intervals.
10. Define frequency polygon. How do you construct a frequency polygon?
11. What do you mean by frequency curve? Mention one advantage of frequency curve.
12. Describe cumulative frequency curves or ogives.
13. Mention one difference between "less than" cumulative frequency curve and "more than" cumulative frequency curve. How can you obtain the median in an ogive?
14. Explain the significance of ogives.
15. Consumers were polled about their favourite ice-cream flavours in a survey. Draw a bar diagram for the following data :

Flavour of ice-cream	Frequency
Vanilla	16
Strawberry	5
Chocolate	12
Mint chocolate	3
Others	6

16. Prepare a multiple bar diagram of the following data :

Faculty	Number of students		
	2014-15	2015-16	2016-17
Arts	600	550	500
Science	400	500	600
Commerce	200	250	300

17. A person spends his time on different activities daily (in hours)

Activity	Office	Exercise	Travelling	Watching	Sleeping	Miscellaneous
Number of hours	9	1	2	3	7	2

Draw a pie chart for this information.

18. Draw a histogram for the following data distribution :

Class intervals	50 - 60	60 - 70	70 -80	80 - 90	90 - 100	100 - 110
Frequency	30	25	45	15	20	40

19. Draw a histogram and a frequency polygon from the following data :

Wages in (Rs)	500 - 600	600 -700	700 -800	800 - 900	900 - 100
No. of workers	15	37	52	26	10

20. Draw the ogives for the following frequency distribution of marks of 100 students in a class.

Marks	20 - 30	30 - 40	40 -50	50 - 60	60 - 70	70 - 80	80-90	90-100
No. of students	7	11	24	32	9	14	2	1

Unit III :
Measures of Central Tendency

III.1 Introduction :

For any statistical investigation, an investigator begins with collection of data is numerical facts. The data so collected are called raw materials. It is from these raw materials the investigators analysis the data after proper classification and tabulation. For analysing the data, we cannot deal with the whole series of data. There is a necessity for some single measurement which may give the summary description of the characteristics of a large group of variables. This single value represent the middle most value of the lowest and highest value of the tabulated values. Measures of central tendency popularly known as AVERAGES serve this purpose. An average represents the entire statistical tabulated data which lies somewhere between the largest and smallest observation and generally it is located in the centre of the distribution.

III.2 Definition of Average :

A measure of central tendency or an average of a certain distribution is a representative value of that distribution which enables us to comprehend in a single effort the significance of the whole. According to the great statistian Clark, "Average value is a single value within the range of the data that is used to represent all the values in the series. An average value lies somewhere in the middle part of the data, it is sometimes called the "Measure of Central Tendency".

Measure of central tendency or averages are also known as measures of central location.

III.3 Characteristics of an Ideal Average :

An average is considered as an ideal average, if it possess the following characteristics–

(i) It should be easy to calculate and easy to understand.

(ii) It should be based on all the observation of series.

(iii) It should be rigidly defined.

(iv) It should not be affected by fluctuations of sampling.

(v) It should not be affected by extreme values.

(vi) It should be capable of further algebraic treatment.

$\sum$ Notion :

The Greek alphabet $\sum$ (Capital sigma) is used to represent sum. For example: If the variable *x* takes the values $x_1, x_2, x_3, ...x_n$ then the sum of these values of *x* is $x_1, x_2, x_3, ...x_n$

denoted by $\sum_{i=1}^{n} x_i$ and it means that lower limit of *i* is 1 and upper limit of *i* is n and *i* takes the values 1, 2, 3, ..., *n*.

Thus, $\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + ... + x_n$

Some rules of $\sum$:

(i) $\sum_{i=1}^{n} (x_i \pm y_i) = \sum_{i=1}^{n} x_i \pm \sum_{i=1}^{n} y_i$

(ii) $\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$

(iii) $\sum_{i=1}^{n} c = nc$

Example 1 : Find the value of (i) $\sum_{i=1}^{8} i$, (ii) $\sum_{i=1}^{5} i^2$

Solution:

(i) $\sum_{i=1}^{8} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$

(ii) $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$= 1 + 4 + 9 + 16 + 25 = 55$

III.4 Different Measures of Central Tendency Or Average :

The different measures of central tendency or average are as follows :

1. Mean, 2. Median, 3. Mode

1. **Mean :** There are three types of mean -
   (a) Arithmetic Mean (A.M.)
   (b) Geometric Mean (G.M.)
   (c) Harmonic mean (H.M.)

1. (a) Arithmetic Mean (A.M.) : The A.M. of a given set of observation is defined to be the sum of values of observations and dividing the total by the number of observation. It is denoted by the symbol $\overline{x}$ of the variable x.

I. A.M. for Discrete or Individual Series :

Let $x_1, x_2, x_3, ..., x_n$ are n observations. Then their A.M. is given by

$$\overline{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Example 2 : Find A.M. of the following -

(i) 3, 7, 9, 11, 15

(ii) –4, –1, 0, 5, 7, 11

Solution:

(i) The required A.M. = $\frac{3+7+9+11+15}{5} = \frac{45}{5} = 9$

(ii) The required A.M. = $\frac{-4-1+0+5+7+11}{6} = \frac{18}{6} = 3$

II. A.M. for Ungrouped or Individual Frequency Distribution :

Let $x_1, x_2, x_3, ..., x_n$ are n values of the variable x and $f_1, f_2, f_3, ..., f_n$ are their respective frequencies.

$$\overline{x} = \frac{f_1 x_1 + f_2 x_2 + f_3 x_3 + ... + f_n x_n}{f_1 + f_2 + f_3 + ... + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \quad \text{where } \sum_{i=1}^{n} f_i = \text{N}$$

$$\overline{x} = \frac{\sum fx}{\text{N}}$$

Example 3 : Find the A.M. of the following distribution :

Age (years)	:	14	15	16	17	18	19	20
No. of students	:	6	9	10	7	5	3	2

Solution :

Table : III.1

Age (x) :	14	15	16	17	18	19	20	Total
Students (f) :	6	9	10	7	5	3	2	N = 42
$f \times x$:	84	135	160	119	90	57	40	$\sum fx = 685$

$$\therefore \text{ AM } (\bar{x}) = \frac{\Sigma(f \times x)}{N} = \frac{685}{42} = 16.30$$

III. Short Cut Method or Assumed Mean Method :

When the values of variable x and their respective frequencies are not large then we can directly apply above formula.

When the values of either x or f or both are large then estimation of Arithmetic mean by this method is time consuming. In such case arithmetic mean is estimated by applying either shortcut method or step-deviation method.

Let the variables x takes the values $x_1, x_2, x_3, ..., x_n$ and let the deviations of these values of x from a constant A (A is called assumed mean of x) be $d_1, d_2, d_3, ..., d_n$ respectively.

$$\therefore\ d_1 = x_i - \text{A};\ i = 1, 2, 3, ...n$$

$$\Rightarrow x_i = d_i + \text{A}$$

(a) In case of Individual Series :

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum_{i=1}^{n}(\text{A} + d_i)}{n} = \frac{\sum_{i=1}^{n}\text{A} + \sum_{i=1}^{n} d_i}{n}$$

$$\bar{x} = \frac{n\text{A}}{n} + \frac{\sum_{i=1}^{n} d_i}{n} = \text{A} + \frac{\Sigma d}{n}$$

(b) In case of frequency distribution :

$$\bar{x} = \text{A} + \frac{\Sigma fd}{\text{N}}$$

IV. Step-Deviation Method :

Let $d' = \frac{d}{h} \Rightarrow d = d'h,$ where $d' = \frac{x - \text{A}}{h}$

(a) In case of individual series :

$$\overline{x} = \text{A} + \frac{\sum d}{h} = \text{A} + \frac{\sum d'h}{n}$$

(b) In case of frequency distribution :

$$\overline{x} = \text{A} + \frac{\sum fd'}{n} \times h$$

Example 4: Using Shortcut or Assumed mean method find A.M.

Weight (kg)	:	30	40	50	60	70	80
No. of Persons	:	6	10	16	8	6	4

Solution : Let weight of person be *x* and number of persons be *f*

Table : III.2

Weight (kg) (*x*)	No. of Persons (*f*)	$d = x - \text{A} = x - 50$	*fd*
30	6	–20	– 120
40	10	– 10	– 100
50	16	0	0
60	8	10	80
70	6	20	120
80	4	30	120
	$\sum f = \text{N} = 50$		$\sum fd = 100$

Let assumed mean, A = 50

$$\overline{x} = \text{A} + \frac{\sum fd}{\text{N}} = 50 + \frac{100}{50} = 50 + 2$$

$$= 52 \text{ kg}$$

Example 5: Using step-deviation method find A.M. of the following :

Wages (Rs.)	:	0-10	10-20	20-30	30-40	40-50
No. of workers	:	15	28	37	23	17

Solution : Let wages be *x* and no, of workers be *f*

Table : III.3

Wages (x)	Mid value (x)	Frequency (f)	$d = x - \text{A}$ $= x - 25$	$d' = \frac{4}{10}$	fd'
0-10	5	15	– 20	–2	– 30
10-20	15	28	– 10	– 1	– 28
20-30	25	37	0	0	0
30-40	35	23	10	1	23
40-50	45	17	20	2	34
		N = 120			$\sum fd' = -1$

Let, A = 25

$$\bar{x} = \text{A} + \frac{\sum fd'}{\text{N}} \times h$$

$$= 25 + \frac{-1}{120} \times 10 = 25 - 0.83 = 24.17$$

V. Weighted Average :

The quantity of a series which are relatively more or less than one another quantity is called weight.

The average obtained by giving due weight to the quantities of a series is called weighted average of the series.

Let the weights attached to the quantities $x_1, x_2, x_3, ..., x_n$ be $w_1, w_2, w_3, ..., w_n$ respectively. The weighted mean of these quantities is given by -

$$\bar{x}_w = \frac{w_1x_1 + w_2x_2 + w_3x_3 + ... + w_nx_n}{w_1 + w_2 + w_3 + ... + w_n} = \frac{\sum wx}{\sum w}$$

Uses of weighted mean :

(i) Weighted mean is used for the series whose weight of each quantity is not equal.

(ii) Weighted mean is used to find mean of the mean of subseries of the series.

(iii) Weighted mean is used to find the average value of ratio and percentage.

Example 6: The marks obtained by three students A, B and C in Accountancy, Business Studies and Banking out of 100 in each subject in a test are as follows :

		Accountancy	Business Studies	Banking
A	:	50	60	70
B	:	65	40	60
C	:	60	50	45

Rank the students if weights are given as below :

40%, 20%, 30%

Solution : Let, *x, y* and *z* denote the marks obtained by A, B and C in Accountancy, Business Studies and Banking respectively.

Let $\bar{x}, \bar{y}$ and $\bar{z}$ denote the weighted averages marks obtained by A, B and C resepctively.

$$\bar{x} = \frac{50 \times 40\% + 60 \times 20\% + 70 \times 30\%}{40\% + 20\% + 30\%}$$

$$= \frac{20 + 12 + 21}{0.4 + 0.2 + 0.3}$$

$$= \frac{53}{0.9} = \frac{530}{9} = 58.88$$

$$\bar{y} = \frac{65 \times 40\% + 40 \times 20\% + 60 \times 30}{40\% + 20\% + 30\%}$$

$$= \frac{26 + 8 + 18}{0.4 + 0.2 + 0.3}$$

$$= \frac{52}{0.9} = \frac{52}{9} = 57.77$$

$$\bar{z} = \frac{60 \times 40\% + 50 \times 20\% + 45 \times 30\%}{40\% + 20\% + 30\%}$$

$$= \frac{24 + 10 + 13.5}{0.4 + 0.2 + 0.3}$$

$$= \frac{47.5}{0.9} = \frac{475}{9} = 52.77$$

From above we find that the ranking position of A, B and C become 1st, 2nd and 3rd respectively.

Properties of A. M. :

1. The algebraic sum of deviations of a set of values from their A.M. is zero.

Proof: Let the values of a variable *x* are $x_1, x_2, x_3, ..., x_n$ and their corresponding frequencies are $f_1, f_2, f_3, ..., f_n$

We are to prove that -

(i) $\Sigma(x_i - \bar{x}) = 0,$ for individual data.

(ii) $\Sigma f_i(x_i - \bar{x}) = 0,$ for grouped data.

(i) Now, L.H.S. = $\Sigma(x_i - \bar{x})$ $\left[\bar{x} = \text{A.M.} = \frac{\Sigma fx}{\text{N}}; \text{N} = \Sigma f\right]$

$$= \Sigma x_i - \Sigma \bar{x}$$

$$= n.\frac{1}{n}\Sigma x_i - n.\bar{x} \because \sum_{i=1}^{n} c = n.c$$

$$= n.\bar{x} - n.\bar{x} = 0$$

(ii) L.H.S. $= \Sigma f_i(x_i - \bar{x})$

$$= \Sigma f_i x_i - \Sigma f_i \bar{x}$$

$$= \text{N}.\frac{1}{\text{N}}\Sigma f_i x_i - \bar{x}.\text{N}$$

$$= \text{N}.\bar{x} - \bar{x}.\text{N} = 0 \quad [\because \Sigma f_i = \text{N}]$$

$$= \text{N}.\bar{x} - \text{N}.\bar{x} = 0$$

2. If the relation between two variables x and y is $y = a + bx,$ whee a and b are constants then $\bar{y} = a + b\bar{x}$

Proof: $y = a + bx$

Taking $\sum$ both side,

$$\Sigma y = \Sigma(a + bx)$$

$$\Rightarrow \frac{\Sigma y}{n} = \frac{\Sigma a}{n} + \frac{b\Sigma x}{n}$$ [dividing both side by n]

$$\Rightarrow \bar{y} = \frac{na}{n} + b\bar{x} \Rightarrow \bar{y} = a + b\bar{x}$$

3. The sum of the squares of deviations of a variable about mean is the least. Mathematically, $\Sigma(x_i - A)^2$ is minimum when $A = \bar{x},$ where A is any arbitrary value.

Proof: Let S be the sum of squares of the deviations of the values from any arbitrary value.

$$S = \Sigma(x_i - A)^2$$

$$= \Sigma(x_i - \bar{x} + \bar{x} - A)^2$$

$$= \Sigma\left[(x_i - \bar{x})^2 + (\bar{x} - A)^2 + 2(x_i - \bar{x})(\bar{x} - A)\right]$$

$$= \Sigma(x_i - \bar{x})^2 + \Sigma(\bar{x} - A)^2 + 0$$

$$= \Sigma(x_i - \bar{x})^2 + n(\bar{x} - A)^2$$

The value of S will be the least when $n(\bar{x} - A)^2$ becomes the least is zero.

$$\therefore\ n(\bar{x} - A)^2 = 0$$

$$\Rightarrow \bar{x} - A = 0,\ \bar{x} = A$$

4. If n_1 and n_2 are number of observations of two series of data and $\bar{x}_1$ and $\bar{x}_2$ are their respective mean of the two series then

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

5. If all the observations of a series are added, subtracted, multiplied or divided by a constant, the mean is also added, subtracted, multiplied or divided by the same constant. For example: The mean of 20 observations is 60. If each observation is increased by 5 then the new mean will be 60 + 5 = 65. If each observation is multiplied by 3, then the mean also multiplied by 3; and the new mean is $60 \times 3 = 180$.

Example 7 : In a competitive test in the subject mathematics 200 candidates appeared of whom 120 were boys and 80 were girls. The mean score of boys is 60 and that of girls is 40. Find the mean score of the 200 candidates.

Solution :

Let the number of boys $= n_1 = 120$

Number of girls $= n_2 = 80$

and let mean score of boys $= \bar{x}_1 = 60$

Mean score of girls $= \bar{x}_2 = 40$

The mean score of the 200 candidates,

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{120 \times 60 + 80 \times 40}{120 + 80}$$

$$= \frac{7200 + 3200}{200}$$

$$= \frac{1040\cancel{0}\cancel{0}}{2\cancel{0}\cancel{0}} = 52$$

Example 8 : In Higher Secondary Examination of Business Mathematics and Statistics average marks obtained by boys was 70 and that of girls was 60. The mean marks of all the students was 65. Find the ratio of the boys and girls and the percentage of boys and girls.

Solution :

Let the number of boys $= n_1$

The number of girls $= n_2$

Average marks of boys $= \bar{x}_1 = 70$

Average marks of girls $= \bar{x}_2 = 60$

Average marks of boys and girls $= \bar{x} = 65$

$$\therefore \bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$\Rightarrow 65 = \frac{n_1 \times 70 + n_2 \times 60}{n_1 + n_2}$$

$$\Rightarrow 65(n_1 + n_2) = 70n_1 + 60n_2$$

$$\Rightarrow 65n_1 - 70n_1 = 60n_2 - 65n_2$$

$$\Rightarrow -5n_1 = -5n_2$$

$$\Rightarrow \frac{n_1}{n_2} = \frac{5}{5}$$

$$\Rightarrow \frac{n_1}{n_2} = \frac{1}{1}$$

$$\therefore n_1 : n_2 = 1:1$$

$\therefore$ Number of boys $= \frac{1}{1+1} \times 100\% = 50\%$

Number of girls $= \frac{1}{1+1} \times 100\% = 50\%$

Example 9 : The average marks in mathematics of 100 students in a class is 70. The average marks of 60 boys is 64. Find average marks of girls in the class.

Solution : Let number of boys $= n_1 = 60$

Number of girls $= n_2$

Total students $= 100$

$\therefore n_1 + n_2 = 100$

$\Rightarrow 60 + n_2 = 100$

$\Rightarrow n_2 = 100 - 60$

$\Rightarrow n_2 = 40$

Let, $\bar{x}_1 =$ average marks of boys = 64

$\bar{x} =$ average marks of students = 70

$\bar{x}_2 =$ average marks of girls

We have, $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$$\Rightarrow 70 = \frac{60 \times 64 + 40 \times \bar{x}_2}{100}$$

$$\Rightarrow 70 \times 100 = 3840 + 40\bar{x}_2$$

$$\Rightarrow 7000 - 3840 = 40\bar{x}_2$$

$$\Rightarrow 3160 = 40\bar{x}_2$$

$$\Rightarrow 40\bar{x}_2 = 3160$$

$$\Rightarrow \bar{x}_2 = \frac{316\cancel{0}}{4\cancel{0}} = 79$$

$\therefore$ Average marks of girls = 79.

Example 10 : The relation between the two variables x and y is $4x - 5y + 8 = 0$ and average value of y is 4, then find the average of x.

Solution :

$4x - 5y + 8 = 0$

$\Rightarrow 4x + 8 = 5y$

$\Rightarrow 5y = 4x + 8$

$\Rightarrow 5\overline{y} = 4\overline{x} + 8$ where $\overline{x}$ and $\overline{y}$ are their average value

$\Rightarrow 5 \times 4 = 4\overline{x} + 8$

$\Rightarrow 20 - 8 = 4\overline{x}$

$\Rightarrow 4\overline{x} = 12$

$$\Rightarrow \overline{x} = \frac{12}{4} = 3$$

$\Rightarrow \overline{x} = 3$

Advantages (Merits) and Disadvantages (Demerits) of A.M. :

Advantages :

(i) A.M. is easy to understand and easy to calculate.
(ii) A.M. is based on all the observations of the distribution.
(iii) The formula for A.M. is rigidly defined.
(iv) A.M. is least affected by the fluctuation of sampling.
(v) A.M. is the best measure to compare two or more series.
(vi) A.M. is suitable for further mathematical treatment.

Disadvantages :

(i) It cannot be determined by inspection.
(ii) A.M. cannot be measured for open end class interval.
(iii) A.M. is very much affected by extreme observation.
(iv) A.M. cannot be used for qualitative characteristics like beauty, honesty, etc.

Uses of A.M. :

(i) Common people uses A.M. for different average values.
(ii) Businessman uses A.M. to find average cost, average profit etc.
(iii) A.M. is used in practical statistics.

Problems Related to Some Special Forms :

Case I : Frequency distribution having open-ended class interval :

Example 11 :

Marks	No. of Students
0 and above	3
10 and above	4
20 and above	13

30 and above	10
40 and above	7
50 and above	3

Find A.M. of the distribution

Solution : In the problem, one end of the class is not mentioned, e.g. 0 and above, 10 and above etc. Such classes are known as open class interval. First we will convert the open class into closed class interval.

Table : III.4

Classs	Mid value (x)	Frequency (f)	fx
0 - 10	5	3	15
10 - 20	15	4	60
20-30	25	13	325
30 - 40	35	10	350
40 - 50	45	7	315
50 - 60	55	3	165
		$\Sigma f = N = 40$	$\Sigma fx = 1230$

$$\therefore \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{1230}{40} = 30.75$$

Case II : When prequencies are given in cumulative form :

Example 12: Calculate A.M. from the following :

Age below (years) :	10	20	30	40	50	60	70	80
No. of persons dying :	15	30	53	75	100	110	115	125

Solution : First we convert the cumulative frequency to frequency and individual value to class interval.

Table : III.5

Age below (in years)	Class Interval (In years)	No. of persons dying	Frequency (f)	Mid-value (x)	fx
10	0-10	15	15	5	75
20	10-20	30	30-15=15	15	225
30	20-30	53	53-30=23	25	575
40	30-40	75	75-53=22	35	770
50	40-50	100	100-75=25	45	1125

60	50-60	110	$110\text{-}100=10$	55	550
70	60-70	115	$115\text{-}100=5$	65	325
80	70-80	125	$125\text{-}115=10$	75	750
			$\sum f=N=125$		4395

$$\therefore\ A.M.=\bar{x}=\frac{\sum fx}{\sum f}=\frac{4395}{125}=35.16$$

Case III : When one or two frequencies are missing :

Example 13: Find the missing frequency f_1 from the following data when mean is 15.38

x :	10	12	14	16	18	20
f:	2	7	f_1	20	8	5

Solution :

Table : III.6

x	f	fx
10	2	20
12	7	84
14	f_1	$14f_1$
16	20	320
18	8	144
20	5	100
	$\sum f=42+f_1$	$\sum fx=14f_1+668$

We have, $\bar{x}=15.38$

and $$\bar{x}=\frac{\sum fx}{\sum f}=\frac{14f_1+668}{42+f_1}$$

$$\Rightarrow 15.38=\frac{14f_1+668}{42+f_1}$$

$$\Rightarrow 15.38(42+f_1)=14f_1+668$$

$$\Rightarrow 15.38\times 42+15.38f_1=14f_1+668$$

$$\Rightarrow 645.96+15.38f_1-14f_1=668$$

$$\Rightarrow 1.38 f_1 = 668 - 645.96$$

$$\Rightarrow 1.38 f_1 = 22.04$$

$$\Rightarrow f_1 = \frac{22.04}{1.38} = \frac{2204}{138} = 15.97 \sim 16$$

Example 14 : Find the missing frequency f_1 and f_2 if the A.M. is given 50.

Class :	0-20	20-40	40-60	60-80	80-100	Total
Frequencies :	17	f_1	32	f_2	19	120

Solution :

Table : III.7

Classs	Mid value (x)	Frequency (f)	$d = x - A$ $= x - 50$	fd
0 - 20	10	17	–40	– 680
20 - 40	30	f_1	–20	$-20f_1$
40-60	50	32	0	0
60 - 80	70	f_2	20	$20f_2$
80 - 100	90	19	40	760
		$\sum f = N = 68 + f_1 + f_2$		$\sum fd = 80 + 20f_2 - 20f_1$

We have, $\bar{x} = A + \frac{\sum fd}{N}$ .... (1)

Given that $\bar{x} = 50$, $d = x - A$, A = Assumed Mean

$$A = 50$$

$$\sum f = N = 120$$

$$\therefore 68 + f_1 + f_2 = \sum f = 120$$

$$\Rightarrow f_1 + f_2 = 120 - 68$$

$$\Rightarrow f_1 + f_2 = 52 \text{ .... (2)}$$

$$(1) \Rightarrow \bar{x} = A + \frac{\sum fd}{N}$$

$$\Rightarrow 50 = 50 + \frac{80 + 20f_2 - 20f_1}{68 + f_1 + f_2}$$

$$\Rightarrow 50-50=\frac{80+20f_2-20f_1}{68+f_1+f_2}$$

$$\Rightarrow 0=\frac{80+20f_2-20f_1}{68+f_1+f_2}$$

$$\Rightarrow 0=80+20f_2-20f$$

$$\Rightarrow 20f_1-20f_2=80$$

$$\Rightarrow f_1-f_2=4 \text{ .... } (3)$$

$$\{(2)+(3)\}\Rightarrow 2f_1=56$$

$$\Rightarrow f_1=\frac{56}{2}$$

$$\Rightarrow f_1=28$$

From (2), $f_1+f_2=52$

$$\Rightarrow 28+f_2=52$$

$$\Rightarrow f_2=52-28=24$$

Case IV : When class intervals are given in overlapping form :

In this case overlapping classes are converted to non-overlapping classes. Then $\bar{x}$ is found in the usual manner as discussed earlier.

Example 15 : Find the mean weekly wages :

Wages (Rs.) :	30-40	30-50	30-60	30-70	30-80	30-90
No. of workers :	8	28	68	86	96	100

Solution :

Calculation of A.M. after converting into non-overlapping class

Table : III.8

Age in years	Age in years	Frequency (cf)	Frequency (f)	Mid-value (x)	$d=x-A$ $=x-55$	$d'=\frac{d}{h}$	fd'
30-40	30-40	8	8	35	–20	–2	–16
30-50	40-50	28	20 (28 - 8)	45	–10	–1	–20
30-60	50-60	68	40	55	0	0	0
30-70	60-70	86	18	65	10	1	18

30-80	70-80	96	10	75	20	2	20
30-90	80-90	100	4	85	30	3	12
			$\Sigma f = N = 100$				$\Sigma fd' = 14$

Let, A = 55, h = 10

$$\therefore \bar{x} = A + \frac{\Sigma fd'}{N} \times h$$

$$= 55 + \frac{14}{100} \times 10$$

$$= 55 + \frac{140}{100}$$

$$= 55 + 1.4 = 56.4$$

1. (b) Geometric Mean (G.M.)

(i) Geometric Mean of Discrete Non-Frequency Distribution or Unweighted :

Let the variable x takes the values $x_1, x_2, x_3, ..., x_n$. The nth root of the product of these n values is called the geometric mean of x and it is denoted by the letter G or G.M.

$$\therefore\ G = (x_1 \times x_2 \times x_3 \times ..... \times x_n)^{\frac{1}{n}}$$

Taking log to both side,

$$\log G = \frac{1}{n}[\log x_1 + \log x_2 + \log x_3 + ... + \log x_n] = \frac{1}{n}[\Sigma \log x_i]$$

$$\therefore G = Anti\log\left[\frac{1}{n}\{\Sigma \log x_i\}\right]$$

Example 16 : Find G.M. of 3, 6, 24 and 48.

Solution : It has 4 quantities

$$\therefore G.M. = (3 \times 6 \times 24 \times 48)^{\frac{1}{4}}$$

$$= \left(3 \times 3 \times 2 \times 3 \times 2^3 \times 3 \times 2^4\right)^{\frac{1}{4}}$$

$$= \left(3^4 \times 2^4 \times 2^4\right)^{\frac{1}{4}}$$

$$= 3^{4\times\frac{1}{4}} \times 2^{4\times\frac{1}{4}} \times 2^{4\times\frac{1}{4}}$$

$$= 3 \times 2 \times 2 = 12$$

(ii) Geometric Mean of Discrete Frequency Distribution or Weighted :

Let $f_1, f_2, f_3, ..., f_n$ are respective frequency of $x_1, x_2, x_3, ..., x_n$, then

$$G.M. = \left(x_1^{f_n} \times x_2^{f_2} \times x_3^{f_3} \times .... \times x_n^{f_n}\right)^{\frac{1}{N}} \text{ where } \mathrm{N} = \Sigma f$$

$$G.M. = \sqrt[N]{x_1^{f_1} \times x_2^{f_2} \times x_3^{f_3} \times .... \times x_n^{f_n}}$$

$$G.M. = \left(x_1^{f_1} \times x_2^{f_2} \times x_3^{f_3} \times .... \times x_n^{f_n}\right)^{\frac{1}{N}}$$

Taking log to both side; we have

$$\log G.M. = \frac{1}{N}\left[\log x_1^{f_1} + \log x_2^{f_2} + \log x_3^{f_3} + ... + \log x_n^{f_n}\right]$$

$$\log G.M. = \frac{1}{N}\left[f_1 \log x_1 + f_2 \log x_2 + f_3 \log x_3 + ... + f_n \log x_n\right]$$

$$\log G.M. = \frac{1}{N}\left[\Sigma f_i \log x_i\right]$$

$$\Rightarrow G.M. = Anti \log\left[\frac{1}{N}\Sigma\left(f_i \log x_i\right)\right]$$

If we write $f_i = w_i$ and $N = \Sigma w_i$; then weighted G.M. = Antilog $\left[\dfrac{\sum_{i=1}^{n}\left(w_i \log x_i\right)}{\sum_{i=1}^{n} w_i}\right]$

Example 17 : Find G.M. of the following distribution :

Marks :	0-10	10-20	20-30	30-40	40-50
Students :	3	7	10	6	2

Solution :

Table : III.9

Mark Classs	Students (f)	Mid-value (x)	log$x$	$f$log$x$
0-10	3	5	0.6990	2.097
10-20	7	15	1.1761	8.2327
20-30	10	25	1.3979	13.979
30-40	6	35	1.5441	9.2646

40-50	2	45	1.6532	3.3064
	$N=\sum f=28$			$\sum f\log x=36.8797$

$$\therefore \log G=\frac{1}{N}\sum f\log x$$

$$=\frac{1}{28}\times 36.8797$$

$$\log G=1.317$$

$$\therefore G=Anti\log(1.317)$$

$$G=20.75$$

Advantages (Merits) and Disadvantages (Demerits) :

Advantages :

(i) G.M. is rigidly defined.

(ii) G.M. is based on all the observations.

(iii) It is suitable for further mathematical treatment.

(iv) It is less affected by extreme fluctuations of sampling.

(v) As compared to A.M., G.M. is less affected by extreme values.

Disadvantages :

(i) G.M. is not easy to calculate and not easy to understand.

(ii) G.M. cannot be deteremined if a series contain at least one negative value.

(iii) If any one of the observations is zero, G.M. becomes zero.

Uses of G.M. :

(i) G.M. is used in the construction of Index number.

(ii) G.M. is used in averaging ratios, rates and percentages.

(iii) Since population increases in geometric progression, the formula for G.M. is used.

1. (c) Harmonic Mean (H.M.) :

The reciprocal of the arithmetic mean of the reciprocals of the values of a distribution is called the harmonic mean.

Harmonic Mean of Discrete Non-Frequency Distribution :

Let, $x_1, x_2, x_3, ..., x_n$ are given set of *n* observations then their H.M. which is denoted by H is given by

$$H=\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+........+\frac{1}{x_n}}=\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}=\frac{n}{\sum\frac{1}{x}}$$

Example 18 : Find H.M. of 5, 10, 15, 20

Solution : H.M. of 5, 10, 15, 20 is

$$H = \frac{4}{\frac{1}{5}+\frac{1}{10}+\frac{1}{15}+\frac{1}{20}} = \frac{4}{0.2+0.1+0.07+0.05}$$

$$= \frac{4}{0.42}$$

$$= \frac{400}{42} = 9.52$$

Harmonic Mean of Discrete Frequency Distribution :

Let $x_1, x_2, x_3, ...x_n$ are n observations and $f_1, f_2, f_3, ...f_n$ are their respective frequencies then their H.M. is defined as -

$$H = \frac{f_1 + f_2 + f_3 + ... + f_n}{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \frac{f_3}{x_3} + ... + \frac{f_n}{x_n}} = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} \frac{f_i}{x_i}} = \frac{\Sigma f}{\Sigma\left(\frac{f}{x}\right)}$$

If we replace $f_i = w_i$; then weighted H.M. = $\dfrac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} \frac{w_i}{x_i}}$

Example 19 : Marks obtained by 20 students are given below. Find Harmonic Mean.

Marks :	0-10	10-20	20-30	30-40	40-50
Students :	2	5	8	3	2

Solution : H.M

Table : III.10

Classs	f	Mid-value (x)	$\frac{1}{x}$	$\frac{f}{x}$
0-10	2	5	$\frac{1}{5} = 0.2$	$2 \times 0.2 = 0.4$
10-20	5	15	$\frac{1}{15} = .067$	$5 \times 0.067 = 0.335$

20-30	8	25	$\frac{1}{25}=.04$	$8\times0.04=0.32$
30-40	3	35	$\frac{1}{35}=.03$	$3\times0.03=0.09$
40-50	2	45	$\frac{1}{45}=.02$	$2\times0.02=0.04$
	$\Sigma f=20$			$\sum\left(\frac{f}{x}\right)=1.185$

$$\therefore H=\frac{\Sigma f}{\Sigma\frac{f}{x}}=\frac{20}{1.185}=16.88$$

Advantages and Disadvantages H.M. :

Advantages :

(i) H.M. is rigidly defined.

(ii) H.M. is based on all observations of the distribution.

(iii) It is suitable for further mathematical treatment.

Disadvantages :

(i) H.M. is is dfficult to calculate and at the same time it is difficult to understand.

(ii) H.M. cannot be deteremined if a series contain at least one zero value.

(iii) H.M. give more weights to the smaller values.

Uses of H.M. :

The harmonic mean is useful in averaging time rates and finding the average price per unit.

Relation Among A.M., G.M. and H.M. :

(a) $A.M.\geq G.M.\geq H.M.$

Proof: Let x_1 and x_2 be two positive quantities and

$$A.M.=\frac{x_1+x_2}{2},\ G.M.=\sqrt{x_1\times x_2},\ H.M.=\frac{2}{\frac{1}{x_1}+\frac{1}{x_2}}$$

Now, $A.M.-G.M.=\frac{x_1+x_2}{2}-\sqrt{x_1\times x_2}$

$$= \frac{\left(\sqrt{x_1}\right)^2 + \left(\sqrt{x_2}\right)^2 - 2\sqrt{x_1 \times x_2}}{2} = \frac{\left(\sqrt{x_1} - \sqrt{x_2}\right)^2}{2}$$

$\because$ Squares of a real quantity is always a positive quantity.

$$\therefore \frac{1}{2}\left(\sqrt{x_1} - \sqrt{x_2}\right)^2 \geq 0$$

$$\Rightarrow A.M. - G.M. \geq 0$$

$$\Rightarrow A.M. \geq G.M. \; .... \; (1)$$

Again,

$$G.M. - H.M. = \sqrt{x_1 \times x_2} - \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}}$$

$$= \sqrt{x_1 \times x_2} - \frac{2}{\frac{x_1 + x_2}{x_1 x_2}}$$

$$= \sqrt{x_1 x_2} - \frac{2x_1 x_2}{x_1 + x_2}$$

$$= \sqrt{x_1 x_2} - \frac{2\sqrt{x_1^2 x_2^2}}{x_1 + x_2}$$

$$= \sqrt{x_1 x_2}\left(1 - \frac{2\sqrt{x_1 x_2}}{x_1 + x_2}\right)$$

$$= \sqrt{x_1 x_2}\left(1 - \frac{x_1 + x_2 - 2\sqrt{x_1 x_2}}{x_1 + x_2}\right)$$

$$= \sqrt{x_1 x_2}\left[\frac{\left(\sqrt{x_1}\right)^2 + \left(\sqrt{x_2}\right)^2 - 2\sqrt{x_1 x_2}}{x_1 + x_2}\right]$$

$$= \sqrt{x_1 x_2}\left[\frac{\left(\sqrt{x_1} - \sqrt{x_2}\right)^2}{x_1 + x_2}\right]$$

Since, $x_1 > 0,\ x_2 > 0,\ x_1x_2 > 0,\ \dfrac{1}{x_1+x_2} > 0$ and $\left(\sqrt{x_1}-\sqrt{x_2}\right)^2 > 0$

$$\therefore \sqrt{x_1x_2}\left[\frac{\left(\sqrt{x_1}-\sqrt{x_2}\right)^2}{x_1+x_2}\right] \geq 0$$

$$\Rightarrow G.M. - H.M. \geq 0$$

$$\Rightarrow G.M. \geq H.M. \text{........ (2)}$$

From (1) and (2), we have $A.M. \geq G.M. \geq H.M.$

(b) For any two positive numbers $A.M. \times H.M. = (G.M.)^2$

Proof: Let x_1 and x_2 be any two positive numbers, then

$$A.M. = \frac{x_1+x_2}{2},\ G.M. = \sqrt{x_1x_2},\ H.M. = \frac{2}{\frac{1}{x_1}+\frac{1}{x_2}}$$

$$\therefore\ A.M. \times H.M. = \frac{x_1+x_2}{2} \times \frac{2}{\frac{1}{x_1}+\frac{1}{x_2}}$$

$$= \frac{x_1+x_2}{2} \times \frac{2}{\frac{x_1+x_2}{x_1x_2}}$$

$$= \frac{x_1+x_2}{2} \times \frac{2x_1x_2}{x_1+x_2}$$

$$A.M. \times H.M. = \frac{2x_1x_2}{2} = x_1x_2$$

$$A.M. \times H.M. = \sqrt{(x_1x_2)^2} = (G.M.)^2$$

$$\therefore A.M. \times H.M. = (G.M.)^2$$

Example 20 : The A.M. and G.M. of two numbers are 7 and 5 respectively, determine the H.M.

Solution: We know that, $A.M. \times H.M. = (G.M.)^2$

$$\Rightarrow H.M.=\frac{(G.M.)^2}{A.M.}$$

Given that $A.M.$ = 7 and $G.M.$ = 5

$$\therefore H.M.=\frac{5^2}{7}=\frac{25}{7}=3.57$$

Example 21 : The G.M. and H.M. of two quantities are respectively 18 and 10.8. Find the two quantities.

Solution: Let the two quantities be x and y

$$\therefore G.M.=\sqrt{xy}=18$$

$$\Rightarrow xy=324 \text{ .... } (1)$$

$$H.M.=\frac{2}{\frac{1}{x}+\frac{1}{y}}=10.8$$

$$\Rightarrow \frac{2}{\frac{x+y}{xy}}=10.8$$

$$\Rightarrow \frac{2xy}{x+y}=10.8$$

$$\Rightarrow \frac{xy}{x+y}=\frac{10.8}{2}=5.4$$

$$\Rightarrow \frac{324}{x+y}=5.4 \text{ [by (1)]}$$

$$\Rightarrow x+y=\frac{324}{5.4}=\frac{3240}{54}=60 \text{ .... } (2)$$

$$(1)\Rightarrow xy=324$$

$$\Rightarrow x(60-x)=324 \text{ [by equation(2)]}$$

$$\Rightarrow 60x-x^2=324$$

$$\Rightarrow x^2-60x+324=0$$

$$\Rightarrow x^2-54x-6x+324=0$$

$$\Rightarrow x(x-54)-6(x-54)=0$$

$$\Rightarrow (x-6)(x-54)=0$$

Either $x-6=0$ or $x-54=0$

$$x=6 \quad x=54$$

From (1); $xy = 324$

When, $x = 6$, $xy = 324$

$$\Rightarrow 6y = 324$$

$$\Rightarrow y = \frac{\cancel{324}}{6} = 54$$

When, $x = 54$, $xy = 324$

$$\Rightarrow 54y = 324$$

$$\Rightarrow y = \frac{\cancel{324}}{54} = 6$$

The two quantities are 6 and 54.

2. Median

Definition :

Median is the middle value of the distribution which divide the distribution into two equal parts. According to Prof. L.R. Connor- "The median is that value of the variable which divides the group into equal parts, one part comprising all values greater and the other, all the values less than the Median."

At the time of determining median, the observations are arranged in ascending or descending order.

2.1 Median of Discrete Non-Frequency Distribution or Individual Series:

Case I : When n is odd :

Let the total number of observations 'n' is odd numbers and the observations are arranged either in ascending or descending order then the $\left(\frac{n+1}{2}\right)$th value from the beginning or from the end will be the median.

Case II : When *n* is even :

If the total number of observations 'n' is even and after arranging the observations either ascending or descending order then the arithmetic mean of the $\frac{n}{2}$th value and the $\left(\frac{n}{2}+1\right)$th value will be the median.

Example 22 : Determine median for the following series :

(i) 44, 43, 42, 40, 45, 49, 48

(ii) 84, 23, 76, 48, 22, 60, 28, 50

Solution :

(i) Arranging the values of the series in ascending order; we get

40, 42, 43, 44, 45, 48, 49

Here $n = 7$ = odd number

$\therefore$ The required median is $\frac{7+1}{2} = 4$ th term = 44

(ii) Arranging the values of the series in ascending order; we get

22, 23, 28, 48, 50, 60, 74, 76, 84

Number of terms $n = 8$

Now, $\frac{n}{2}$th term $= \frac{8}{2} = 4$th term = 48

$\left(\frac{n}{2}+1\right)$th term $= \left(\frac{8}{2}+1\right) = (4+1) = 5$th term = 50

$\therefore$ The required median $= \frac{48+50}{2} = \frac{98}{2} = 49$

Example 23 : Find median of the following data :

Weekly salary (Rs.) :	15	16	17	18	19
No. of workers: 6	6	12	23	30	9

Solution:

Table : III.11

Weekly salary (x)	No. of workers (f)	cf
15	6	6
16	12	18
17	23	41
18	30	71
19	9	80
	$\sum f = 80 = N$	

$N = 80$; even $\frac{N}{2} = \frac{80}{2} = 40$ and $\frac{N}{2} + 1 = 40 + 1 = 41$

From the *c.f.* of the above table, we have 40 and 41 lie between 18 and 71.

Since 17 is the *c.f.* of 41.

Hence each of the 40 and 41st terms will be 17. Hence the required median is 17.

2.2 Median of a Grouped Frequency Distribution :

In order to determine the median of a grouped distribution, we have to find out the median class from the distribution. To locate the median class intervals, following are the necessary steps.

Step I : First prepare less than cumulative frequency distribution.

Step II : If total observation is N, find $\frac{N}{2}$.

Step III : Find nearest *c.f.* just greater than $\frac{N}{2}$.

Step IV : The class which corresponds to this *c.f.* is the median class.

Step V : The value of median is obtained by using the following formula.

$$\text{Median (me)} = \ell + \frac{\frac{N}{2} - c.f}{f} \times h$$

Where, l = Lower limit of median class

$N = \Sigma f$

$c.f.$ = Cumulative frequency of the class preceeding the median class

h = Width of the median class

Calculation of Median for Various Types of Series :

Type I : When cumulative frequency distribution is less than type :

Example 24 : Following are the marks obtained by 200 students in Business Statistics. Find Median.

Marks less than :	10	20	30	40	50	60	70
No. of students :	15	35	60	84	96	127	200

Solution:

Table : III.12

Marks less than	Marks	No. of students	Frequency (*f*)	*c.f.*
10	0-10	15	15	15
20	10-20	35	35–15 = 20	35
30	20-30	60	60–35 = 25	60
40	30-40	84	84 – 60 = 24	84
50	40-50	96	96 – 84 = 12	96
60	50-60	127	127 – 96 = 31	127
70	60-70	200	200 – 127 = 73	200
			$\sum f = N = 200$	

Here, $N = 200$; $\frac{N}{2} = \frac{200}{2} = 100$

$\therefore$ The *c.f.* just higher than 100 is 127.

The median class is 50-60

We know, Median = $1 + \frac{\frac{N}{2} - c.f.}{f} \times h$

$$= 50 + \frac{100 - 96}{31} \times 10$$

$$= 50 + \frac{40}{31} = 50 + 1.29 = 51.29$$

Type II : When cumulative frequency distribution is more than type :

Example 25 : Following are the marks obtained by 80 students in Business Mathematics. Find Media.

Marks above :	0	10	20	30	40	50	60	70	80	90	100
No. of students :	80	77	72	65	55	43	23	16	10	8	0

Solution:

Table : III.13

Marks above	Marks	No. of students	Frequency (*f*)	*c.f.*
10	0-10	80	80 – 77 = 3	3
20	10-20	77	77–72 = 5	8
30	20-30	72	72–65 = 7	15
40	30-40	65	65–55 = 10	25
50	40-50	55	55–43 = 12	37
60	50-60	43	43–23 = 20	57
70	60-70	23	23–16 = 7	64
80	70-80	16	16–10 = 6	70
90	80-90	10	10–8 = 2	72
100	90-100	8	8–0 = 8	72
	100-110	0		80
			$\sum f = N = 80$	

$$\frac{N}{2} = \frac{80}{2} = 40\text{th term}$$

$\therefore$ *c.f.* just greater than 40 is 57.

$\therefore$ Median class is 50-60

$\therefore$ $l = 50$, $c.f. = 37$, $h = 10$, $f = 20$

$$\text{Median} = l + \frac{\frac{N}{2} - c.f.}{f} \times h$$

$$= 50 + \frac{40-37}{20} \times 10$$

$$= 50 + \frac{3}{2}$$

$$= 50 + 1.5 = 51.5$$

Type III : Exclusive type of class interval:

Example 26 : Determine the median for the following distribution.

Marks :	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45
No. of students :	5	6	15	10	5	4	3	2

Solution: Since classes are continuous i.e. 5-10, 10-15 etc. One number is repeated, we should exclude 10 in the first class and so on.

Table : III.14

Marks	No. of students (*f*)	*c.f.*
5-10	5	5
10-15	6	11
15-20	15	26
20-25	10	36
25-30	5	41
30-35	4	45
35-40	3	48
40-45	2	50
	$\sum f = N = 50$	

$$\frac{N}{2} = \frac{50}{2} = 25\text{th term}$$

$\therefore$ *c.f.* just greater than 25 is 26.

$\therefore$ Median class is 15-20

$\therefore l = 15, c.f. = 11, h = 5, f = 15$

$$\text{Median} = l + \frac{\frac{N}{2} - c.f.}{f} \times h$$

$$= 15 + \frac{25 - 11}{15} \times 5$$

$$= 15 + \frac{14}{3}$$

$$= 15 + 4.66 = 19.66$$

Type IV : Inclusive type of class interval:

Example 27 : Find median from the following distribution.

Marks :	30-34	35-39	40-44	45-49	50-54
No. of students :	5	11	26	10	8

Solution: The data is given in inclusive form of class interval as both are included in the class 30-34.

First we have to convert into exclusive form by taking class boundaries.

Table : III.15

Weight (kg)	Class Boundary	No. of persons (*f*)	*c.f.*
30-34	29.5 – 34.5	5	5
35-39	34.5 – 39.5	11	16
40-44	39.5 – 44.5	26	42
45-49	44.5 – 49.5	10	52
50-54	49.5 – 54.5	8	60
		$\sum f = N = 60$	

$$\frac{N}{2}=\frac{60}{2}=30\text{th term}$$

$\therefore$ *c.f.* just greater than 30 is 42.

$\therefore$ Median class is 40-44

$\therefore\ l=40,\ c.f.=16,\ h=5,\ f=26$

$$\text{Median }=l+\frac{\frac{N}{2}-c.f.}{f}\times h$$

$$=40+\frac{30-16}{26}\times 5$$

$$=40+\frac{14\times 5}{26}$$

$$=40+\frac{70}{26}$$

$$=40+2.69=42.69$$

Type V : To find missing frequency when median is given.

Example 28 : In the following frequency distribution, two class frequencies f_1 and f_2 are missing but the median value of the given data is 46. Find the missing frequencies f_1 and f_2.

Classes :	10-20	20-30	30-40	40-50	50-60	60-70	70-80	Total
Frequencies :	12	30	f_1	65	f_2	25	18	229

Solution: Given Median = 46; $N = \sum f = 229$

We know that, Median = $l + \dfrac{\frac{N}{2} - c.f.}{f} \times i$ .... (1)

Table : III.16

Classes	Frequencies	$c.f.$
10-20	12	12
20-30	30	42
30-40	f_1	42 + f_1
40-50	65	107 + f_1
50-60	f_2	107 + $f_1 + f_2$
60-70	25	132 + $f_1 + f_2$
70-80	18	150 + $f_1 + f_2$
	$N = \sum f = 150 + f_1 + f_2$	

Now, $150 + f_1 + f_2 = N = 229$

$\Rightarrow f_1 + f_2 = 229 - 150$

$\Rightarrow f_1 + f_2 = 79$ .... (2)

$\because$ Median is 46 which lie 40-50

$\therefore$ Median class is 40-50

$\therefore l = 40,\ f = 65,\ c.f. = 42 + f_1,\ h = 10$

$$(1) \Rightarrow 46 = 40 + \frac{\frac{229}{2} - (42 + f_1)}{65} \times 10$$

$$\Rightarrow 46-40=\frac{\frac{229-84-2f_1}{2}}{65}\times 10$$

$$\Rightarrow 6\times 65=\frac{(229-84-2f_1)}{2}\times 10$$

$$\Rightarrow 390=\frac{(229-84-2f_1)}{2}\times 10$$

$$\Rightarrow 39\times 2=220-84-2f_1$$

$$\Rightarrow 78=145-2f_1$$

$$\Rightarrow 2f_1=145-78$$

$$\Rightarrow 2f_1=67$$

$$\Rightarrow f_1=33.5$$

(2) $\Rightarrow f_1+f_2=79$

$$\Rightarrow 33.5+f_2=79$$

$\Rightarrow f_2=79-33.5=45.5$ $\Sigma f = N = 100$

Type VI : When Mid values are given :

Example 29 : Calculate median from the following :

Mid value :	15	25	35	45	55
Frequencies :	4	10	26	8	2

Solution: In the given problem mid values are given. The difference between two consecutive mid values is d = 10 and therefore $\frac{d}{2}=\frac{10}{2}=5$ have to be subtracted from mid value to get lower limit and $\frac{d}{2}=5$ must be added with mid value to get upper limit. Example 15–5 = 10 lower limit 15 + 5 = 20 is upper limit etc.

Table : III.17

Mid value	Class interval	Frequency	*c.f.*
15	10-20	4	4
25	20-30	10	14
35	30-40	26	30

45 55	40-50 50-60	8 2	38 40
		$\Sigma f = N = 40$	

Here $N = 40$, $\frac{N}{2} = \frac{40}{2} = 20\text{th}$ term

$\therefore$ Median class is 30-40

$\therefore l = 30,\ f = 26,\ c.f. = 14 + f_1,\ h = 10$

$$\therefore \text{ Median } = l + \frac{\frac{N}{2} - c.f.}{f} \times h$$

$$= 30 + \frac{30 - 14}{26} \times 10$$

$$= 30 + \frac{16 \times 10}{26}$$

$$= 30 + \frac{160}{26}$$

$$= 30 + 6.14 = 36.14$$

2.3 Advantages and Disadvantages of Median :

Advantages :

(i) It is easy to calculate and easy to understand.
(ii) It is rigidly defined.
(iii) It is not affected by extreme observations.
(iv) Median is most suitable average for open-end classes.
(v) Median sometimes can be located by simple inspection.
(vi) Median can be calculated for unequal class interval.

Disadvantages :

(i) Median is not based on each and every items of the distribution.
(ii) Median is affected more by fluctuation of sampling.
(iii) Median requires arrangement of data either in ascending or in descending order.
(iv) Median in some cases not capable to further algebraic treatment.

2.4 Uses of Median

(i) Median is used for distribution with open-end classes.
(ii) Median can give best result in case of income distribution.
(iii) Median can be used for observations of qualitative nature like beauty, honesty etc.

2.5 Graphical Method for Determination of Median:

Median can be determined graphically by the following methods :

Method (1) : In this method, by taking the variation on x-axis and the cumulative frequency on y-axis, we will draw less than (or greater than) type cumulative frequency curves or ogives.

From the point of intersection of the two curves (ogives), let us draw perpendicular on the x-axis. The point on the x-axis where the perpendicular meet gives the median.

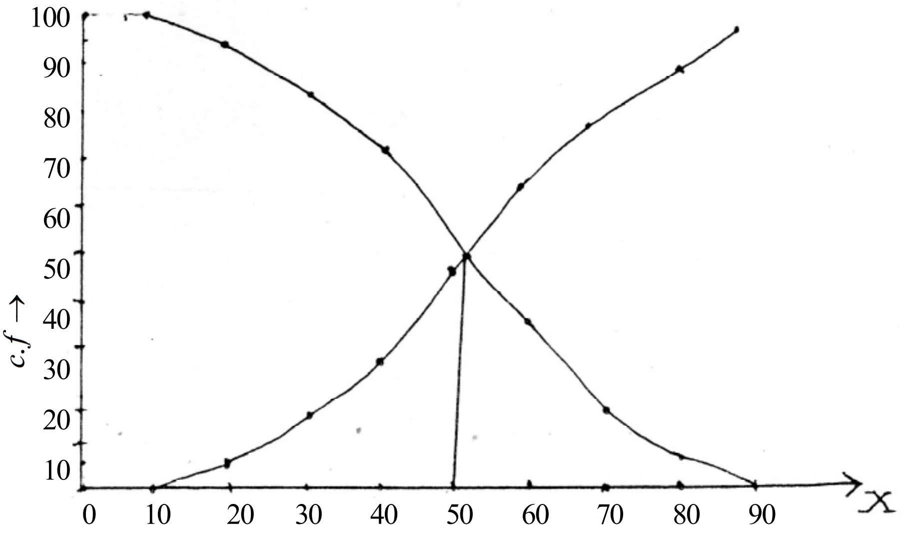
Example 30 : Find the median graphically from the following table :

Daily wages :	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Frequencies :	5	12	13	20	18	15	10	7

Solution: Cumulative frequency distribution table

Table No. 11.25

Daily wages (C.I)	No. of workers (*f*)	c.f. (less than)	*c.f.* (more than)
10-20	5	5	100
20-30	12	17	95
30-40	13	30	83
40-50	20	50	70
50-60	18	68	50
60-70	15	83	32
70-80	10	93	17
80-90	7	100	7
Total	$\Sigma f = N = 100$		



In the above figure, the perpendicular drawn from the point of intersection of the two ogives meet the x-axis at the point 50. So the median is 50.

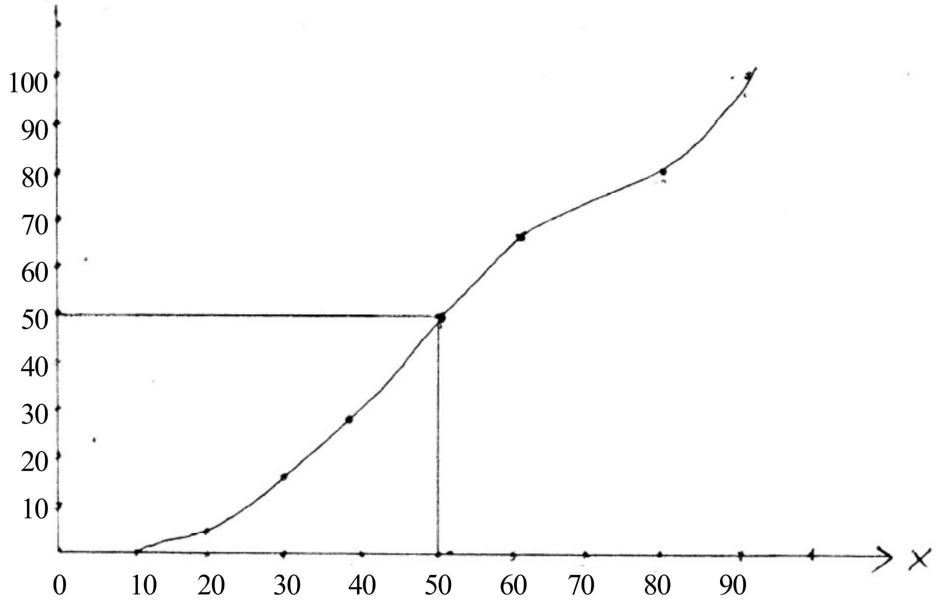
Method (2) : In this method, let us draw a less than (or greater than) type ogive by taking the variation in the x-axis and the cumulative frequency on y-axis. Now taking the $\left(N/2\right)$th observation as median value and locate this value $N/2$ on the y-axis and from it let us draw a perpendicular on the ogive and from the point where the perpendicular line meet the ogive, let us draw a perpendicular on the x-axis and the point where it meets the x-axis is the median.

Example 31 : Fromm the example (30) we will take the table for this example.

Solution: Median = value of $\left(\frac{100}{2}\right)$th item.

= value of 50th item.

Now taking 50 on the y-axis; we draw a horizontal line to meet the ogive. From this point we draw a perpendicular line on x-axis. The point on the x-axis where the perpendicular meet is the median. So the median is 50.



3. Mode :

3.1 Definition :

The mode of a distribution is the observation whose frequency is the maximum. Mode is not unique quantity. That means a distribution may have more than one modes.

3.2 Mode for Discrete Frequency Distribution :

In this case mode is the value of the variable corresponding to the maximum frequency.

Example 30 : Find mode from the following data :

Weekly salary (Rs.) :	15	16	17	18	19	20
No. of workers :	6	12	23	30	9	1

Solution: From the given data, we observe that highest number of workers is 30.

Corresponding to the salary Rs. 18.

$\therefore$ Mode = Salary Rs. 18.

3.3 Mode for Group Frequency Distribution :

Mode of a grouped frequency distribution is obtained by using the following formula :

$$\text{Mode } (M_0) = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

Where, $l =$ lower limit

$f_0 =$ frequency of the class preceeding the model class.

$f_1 =$ frequency of the model class

$f_2 =$ frequency of the class succeeding the model class.

$h =$ length of the model class

Note: (i) The model class is the class whose frequency is highest.

(ii) The above formula is used when all the classes are of equal length

Example 32 : Find the mode of the following data :

Marks :	10-20	20-30	30-40	40-50	50-60	60-70
No. of students :	5	8	12	16	10	8

Solution: From

Table No. III.18

Marks	No. of students
10-20	5
20-30	8
30-40	12 $\rightarrow f_0$
40-50	16 $\rightarrow f_1$
50-60	10 $\rightarrow f_2$
60-70	8

Highest frequency is 16 andn therefore model class is 40-50

$\therefore l = 40,\ f_1 = 16,\ f_0 = 12,\ f_2 = 10,\ h = 10$

$$\therefore \text{ Mode } = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

$$= 40 + \frac{16 - 12}{2 \times 16 - 12 - 10} \times 10$$

$$= 40 + \frac{4 \times 10}{32 - 22}$$

$$= 40 + \frac{40}{10} = 40 + 4 = 44$$

Example 33 : The following data gives the distribution of 100 families according to expenditure. If mode of the distribution is 24, find the missing frequencies *a* and *b*.

Expenditure :	0-10	10-20	20-30	30-40	40-50
Number of families :	14	*a*	27	*b*	15

Solution: From

Table No. III.19

Expenditure	No. of families
0-10	14
10-20	$a \to f_0$
20-30	$27 \to f_1$
30-40	$b \to f_2$
40-50	15
	$\sum f = N = a + b + 56$

$\therefore$ Mode = 24, the model class will be 20-30

and $N = a + b + 56 = 100$

$\Rightarrow a + b = 100 - 56$

$\Rightarrow a + b = 44$ .... (1)

$$\text{Again, Mode } = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

$$\Rightarrow 24 = 20 + \frac{27-b}{2\times 27 - a - b}\times 10$$

$$\Rightarrow 24 - 20 = \frac{27-b}{54 - a - b}\times 10$$

$$\Rightarrow 4 = \frac{27-b}{54-(a+b)}\times 10$$

$$\Rightarrow 4 = \frac{27-b}{54-44}\times 10 \quad \text{[From (1)]}$$

$$\Rightarrow 4 = \frac{27-b}{10}\times 10$$

$$\Rightarrow 4 = 27 - b$$

$$\Rightarrow b = 27 - 4 = 23$$

$$(1) \Rightarrow a + b = 44$$

$$\Rightarrow a + 23 = 44$$

$$\Rightarrow a = 44 - 23 = 21$$

3.5 Merits and Demerits of Mode :

Merits :

(i) Mode can be determined by inspection.

(ii) Mode is easy to calculate and easy to understand.

(iii) Mode is not affected by extreme values.

(iv) Mode may be used to determine qualitative as well as quantative data.

Demerits :

(i) Mode is not suitable for further mathematical treatment.

(ii) Mode is not rigidly defined.

(iii) Mode is not based on all the observation of the distribution.

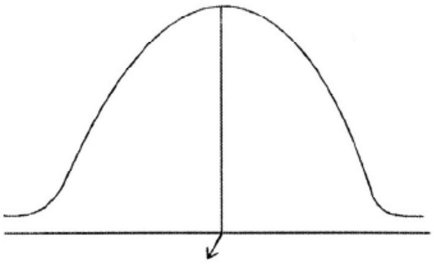
(iv) Mode cannot be determined for unequal class intervals.

3.6 Uses of Mode :

(i) Mode is used as ideal average. For example, in business forecasting like sales of readymade garments etc.

(ii) Mode is useful in the study of maximum consumer preferences.

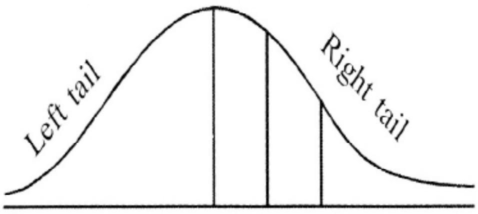
3.7 Relationship among Mean, Median and Mode :



Mean = Mode = Median

Symmetrical Distribution

Fig. III. 1



Mode Mean Median

Shewed Distribution (non-symmetrical)

Fig. III. 2

A distribution where Mean, Median and Mode are equal is known as symmetrical distribution Fig. III.1, otherwise it is known as skewed or non-symmetrical distribution, Fig. III.2.

In the asymmetrical distribution there exist a relationship among Mean, Median and Mode. In the relationship, the distance between Mean and Mode is three times the distance between Mean and Median. That is -

Mean – Mode = 3 (Mean – Median)

We can write the above relation as -

Mode = 3 Median – 2 Mean

This relation is called the Empirical relation

3.8 Graphical method of determination of Mode :

In case of frequency distribution, Mode can be located graphically.

The procedure to locate Mode graphically as follows:

(i) To draw a histogram from the given data, the tallest rectangle of the histogram will represent the model class.

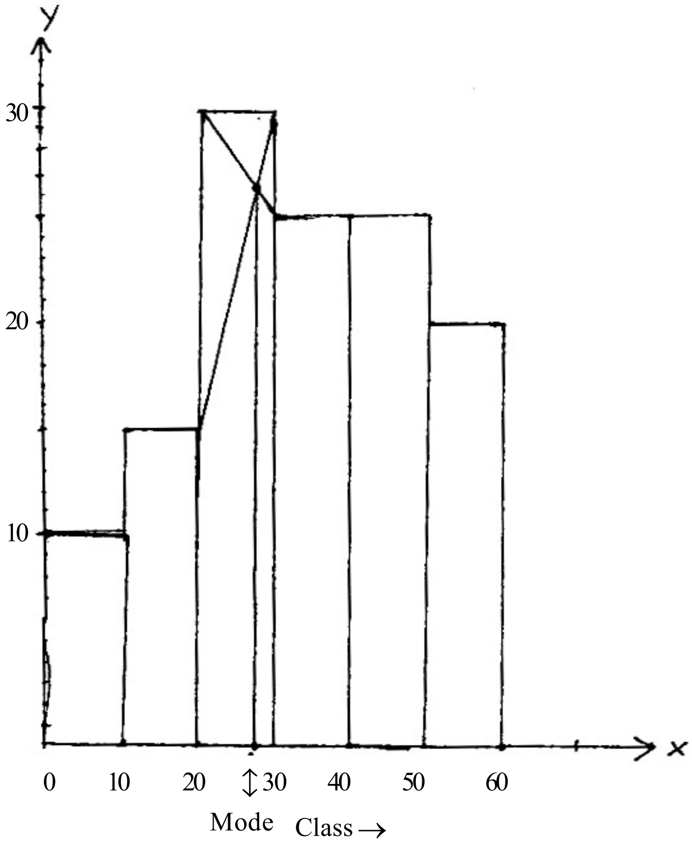
(ii) Two diagonal lines are drawn from top corner of the model class starting each diagonal from top corner of the adjacent rectangle (left & right).

(iii) A perpendicular line is drawn from the point of intersection of these two diagonal lines to x-axis. The value of the point where the perpendicular meets x-axis is the mode.

Example (34) : Find the mode graphically from the following data.

Classes :	0-10	10-20	20-30	30-40	40-50	50-60
Frequency :	10	15	30	25	25	20

Solution:



Example 35 : Calculate mean and median and from the empirical relation find Mode from the following distribution :

Class interval :	50-59	60-69	70-79	80-89	90-99	100-109
Frequency :	4	20	40	50	30	6

Solution:

Table No. III.20

Class Interval	Class boundaries	Mid value (x)	f	$d = x - 74.5$	$d' = \frac{d}{10}$	fd'	$c.f.$
50-59	49.5-59.5	54.5	4	–20	–2	–8	4
60-69	59.5-69.5	64.5	20	–10	–1	–20	24
70-79	69.5-79.5	74.5	40	0	0	0	64

80-89	79.5-89.5	84.5	50	10	1	50	114
90-99	89.5-99.5	94.5	30	20	2	60	144
100-109	99.5-109.5	104.5	6	30	3	18	150
Total			N = 150			$\sum fd' = 100$	

Taking $A = 74.5, h = 10$

1. $A.M = \bar{x} = A + \frac{\sum fd'}{N} \times h$

$$= 74.5 + \frac{100}{150} \times 10$$

$$= 74.5 + 6.6 = 80.5$$

2. Median = Me = $l + \frac{\frac{N}{2} - f_c}{f} \times h$

Now, $\frac{N}{2} = \frac{1}{2} = 75$ nearest to 114

$\therefore$ Median class 79.5-89.5

$\therefore\ l = 79.5\ f_c = 64,\ f = 54,\ h = 10$

$\therefore$ Me = $79.5 + \frac{75 - 64}{50} \times 10$

$$= 79.5 + \frac{11}{5}$$

$$= 79.5 + 2.2 = 81.7$$

The empirical relation is, Mean – Mode = 3 (Mean – Median)

$\Rightarrow$ Mode = 3 Median – 2 Mean

$\Rightarrow$ Mode $= 3 \times 81.7 - 2 \times 80.5$

$\Rightarrow$ Mode $= 245.1 - 161.0 = 84.1$

III.5 Partition Values :

In finding average value of a series of statistical data, median divide the series into two equal parts.

There are some positional values that divide the series more than two equal or unequal parts. These dividing values are known as partition values.

When one point divides a series into two equal parts called halves or median, similarly three points divide a series into four equal parts called Quartile, nine points (9) divide it into ten equal parts called Deciles and ninety-nine points divide it into one hundred equal parts called percentiles. All these values Quartiles, Deciles and Percentiles are called partition values.

III.5.1 Quartiles :

When quantities of a distribution are divided into four equal parts by three points, each part is called Quartile. The three points first,second and third are respectively known as first quartile, second quartile and third quartile and are denoted Q_1, Q_2 and Q_3 respectively.

III.5.1 (a) Quartiles for Discrete non-Frequency Series or Individual Series :

Let $x_1, x_2, x_3, ..., x_n$ are n values of the variable x. The Quartile of these values are given by

$$Q_1 = \text{value of } i\left(\frac{n+1}{4}\right)th \text{ item, } i = 1, 2, 3$$

$$\text{when } i = 1,\ Q_1 = \frac{n+1}{4} \qquad Q_i\,(i = 1,2,3)$$

$$\text{when } i = 2,\ Q_2 = \frac{2(n+1)}{4}$$

$$\text{when i} = 3,\ Q_3 = \frac{3(n+1)}{4}$$

Note: To determine quartile for Individual series, first arrange the given data in ascending order of magnitude and find the number of observation and then use the above formula.

Example 36 : The heights (in cm) of few students are as follows :

70, 72, 71, 75, 69, 73, 74, 76, 75, 70

Solution: Arranging in ascending order, 69, 70, 70, 71, 72, 73, 74, 75, 75, 76

Here, $n = 10$

$$\therefore\ Q_1 = \text{ value of }\left(\frac{n+1}{4}\right) = \frac{10+1}{4} = \frac{11}{4} = 2.75 \text{th observation}$$

$$= \text{2nd observation} + 0.75 \times (\text{3rd observation} - \text{2nd observation})$$

$$= 70 + 0.75 \times (70 - 70)$$

$= 70 + 0.75 \times 0 = 70$

$Q_2 =$ value of $\frac{2(n+1)}{4} = \frac{2\times 11}{4} = \frac{22}{4} = 5.5$ th observation

$=$ 5th observation $+ 0.5 \times$ (6th observation – 5th observation)

$= 72 + 0.5 \times (73 - 72)$

$= 72 + 0.5 \times 1 = 72.5$

$Q_3 =$ value of $\frac{3(n+1)}{4} = \frac{3\times 11}{4} = \frac{33}{4} = 8.25$ th observation

$=$ 8th observation $+ 0.25 \times$ (9th observation – 8th observation)

$= 75 + 0.25 \times (75 - 75)$

$= 75 + 0.25 \times 0 = 75 + 0 = 75$

III.5.1 (b) Quartiles for Discrete Frequency Distribution :

Let $x_1, x_2, x_3, ..., x_n$ are $'n'$ variables of x with frequency $f_1, f_2, f_3, ... + f_n$ respectively. To find Q$_1$, Q$_2$, Q$_3$ proceed the following steps :

1. First find $\sum f = N$
2. Find *c.f.*
3. Find $\frac{i(N+1)}{4}$, to compute $Q_i (i = 1, 2, 3)$
4. Find *c.f.* which is higher than $\frac{i(N+1)}{4}$

Example 37 : Determine the Quartiles from the following distribution :

Ages (yrs) :	50	52	54	58	60	62	64	66	68	70
Frequency :	4	12	18	23	30	26	22	16	5	4

Solution:

Table No. III.21

Ages (yrs) :	50	52	54	58	60	62	64	66	68	70	
Frequency (*f*) :	4	12	18	23	30	26	22	16	5	4	$\sum f = N = 160$
c.f. :	4	16	34	57	87	113	135	151	156	160	

$$Q_1 = \frac{N+1}{4} = \frac{160+1}{4} = \frac{161}{4} = 40.25\text{th} \text{ item.}$$

Nearest to 40.25th item of *c.f.* is 57.

Corresponds to 57 is 58

$$\therefore Q_1 = 58 \text{ years}$$

$$Q_2 = \frac{2(N+1)}{4} = \frac{2(160+1)}{4} = \frac{2\times 161}{4} = \frac{322}{4} = 80.5\text{th}$$

Nearest to 80.25th item of *c.f.* is 87.

Corresponds to 87 is 60

$$\therefore Q_2 = 60 \text{ years}$$

$$Q_3 = \frac{3(N+1)}{4} = \frac{3(160+1)}{4} = \frac{3\times 161}{4} = \frac{483}{4} = 120.75\text{th}$$

Nearest to 120.75th item of *c.f.* is 135.

Corresponds to 135 is 64

$$\therefore Q_3 = 64 \text{ years}$$

III.5.1 (c) Quartiles for Continuous Series or Grouped Frequency Distribution :

To find Quartile, following are the steps

Step I : Find $\sum f = N$ and then find $\frac{iN}{4}(i = 1, 2, 3)$

Step II : Find cumulative frequency which is just greater than $\frac{iN}{4}$ and the class corresponds to this c.f. is the corresponding with Quartile class.

Step III : Calculate $Q_i\ (i = 1, 2, 3)$ by the following formula :

$$Q_i = l + \frac{\frac{iN}{4} - c.f.}{f} \times h(i = 1, 2, 3)$$

Where l = lower limit of the particular Quartile class

f = frequency of the particular Quartile class

h = width of the Quartile class

$c.f.$ = cumulative frequency

$N = \sum f$

Example 38 : Calculate different Quartile from the following table :

Salary (Rs.) :	4-8	8-12	12-16	16-20	20-24	24-28	28-32	32-36	36-40
No. of workers:	6	10	18	30	15	12	10	5	2

Solution:

Table No. III.22

Salary (Rs.) :	4-8	8-12	12-16	16-20	20-24	24-28	28-32	32-36	36-40	Total
No.of workers (f)	6	10	18	30	15	12	10	5	2	108
Mid value (x)	6	10	14	18	22	26	30	34	38	
$c.f.$:	6	16	34	64	79	91	101	106	108	

For $Q_1, \frac{N}{4} = \frac{108}{4} = 27$ nearest to $c.f.$ 34

$\therefore$ Q_1 class is 12-16

$\therefore$ $l = 12, c.f. = 16, f = 18, h = 4$

$$Q_1 = l + \frac{\frac{N}{4} - c.f.}{f} \times h$$

$$= 12 + \frac{27 - 16}{18} \times 4$$

$$= 12 + \frac{11 \times 4}{18}$$

$$= 12 + \frac{11 \times 2}{9}$$

$$= 12 + 2.44 = 14.44$$

For $Q_2, \frac{2N}{4} = \frac{2 \times 108}{4} = 2 \times 27 = 54$ nearest to 64

$\therefore$ Q_2 class is 16-20

$\therefore$ $l = 16, c.f. = 34, f = 30, h = 4$

$$Q_2 = l + \frac{\frac{2N}{4} - c.f.}{f} \times h$$

$$= 16 + \frac{54 - 34}{30} \times 4$$

$$= 16 + \frac{20 \times 4}{30} = 18.66$$

For $Q_3, \frac{3N}{4} = \frac{3 \times 108}{4} = 3 \times 27 = 81$ nearest to 91

$\therefore$ Q_3 class is 24-28

$\therefore\ l = 24, c.f. = 79, f = 12, h = 4$

$$Q_3 = l + \frac{\frac{3N}{4} - c.f.}{f} \times h$$

$$= 24 + \frac{81 - 79}{12} \times 4$$

$$= 24 + \frac{2}{3}$$

$$= 24 + .67 = 24.67$$

Uses of Qurtiles :

Some of the uses of Quartiles are :

(i) Quartiles are used to measure central tendency as 2nd Quartile is the median.

(ii) Q_1 and Q_3 are used to measure dispersion as because $(Q_3 - Q_1)/2$ is Quartile deviation.

(iii) Quartiles are used to measure skewness as $\frac{Q_3 - Q_1 - 2Q_2}{Q_3 - Q_1}$ gives skewness.

III.5.2 Deciles:

Deciles are those values which divide the given distributions into 10 equal parts by 9 (nine) points. Deciles are denoted by D_1, D_2, D_3, ..., D_9.

III.5.2 (a) : Deciles for Individual Observation:

Let $x_1, x_2, x_3, ..., x_n$ are *n* values of the variables *x*. Then deciles of these values are given by

$$D_i = \frac{i(n+1)}{10};\ (i = 1,2,3,......,8,9)$$

First decile, $D_1 = \frac{n+1}{10}$

Second decile, $D_2 = \frac{2(n+1)}{10}$ and so on.

III.5.2 (b) : Deciles for Continuous Frequency Distribution or Grouped Frequency Observations:

The deciles for grouped frequency distributions are given by -

$$D_i = l + \frac{\frac{iN}{10} - c.f.}{f} \times h \quad (i = 1, 2, 3, ..., 9)$$

Where $N = \Sigma f$

l = lower limit of particular decile class

$c.f.$ = cumulative frequency

f = frequency of decile class

h = width of the class

III.5.3 Percentiles :

Percentiles are those values which divide the given distributions into hundred equal parts by 99 points. Percentiles are denoted by P_1, P_2, P_3, P_4, ..., P_{99}.

III.5.3 (a) Percentiles for Individual Observation :

Let $x_1, x_2, x_3, ..., x_n$ are n values of the variables x. Then the percentiles of these values are given by

$$P_i = \frac{i(n+1)}{100};\ (i = 1,2,3,4,.....,99)$$

First percentile, $P_1 = \frac{n+1}{100}$

Second percentile, $P_2 = \frac{2(n+1)}{100}$ and so on.

III.5.3 (b) Percentiles for Continuous Frequency Distribution or Grouped Frequency Distributions :

The percentiles for grouped frequency distributions are given by -

$$P_i = l + \frac{\frac{iN}{100} - c.f.}{f} \times h \quad (i = 1, 2, 3, ..., 99)$$

Where $N = \sum f$

l = lower limit of percentile class

$c.f.$ = cumulative frequency

f = frequency of particular percentile class

h = width of the percentile class

Example 39 : Find Q_1, Q_3, Q_4, D_4, D_6, P_{30}, P_{70} from the following table:

Salary (Rs.) :	10	14	18	22	24	30	32	Total
No. of workers:	5	7	10	16	14	8	4	64

Solution:

Table No. III.23

Salary (Rs.) :	10	14	18	22	24	30	32	Total
No.of workers :	5	7	10	16	14	8	4	64
c.f. :	5	12	22	38	52	60	64	

Here, $N = \sum f = 64$

$\therefore Q_1 = \frac{N+1}{4} = \frac{64+1}{4} = \frac{65}{4} = 16.25\text{th}$ item nearest *c.f.* to 22

$\therefore Q_1$ class is 18

$\therefore Q_1 =$ Rs. 18

$\therefore Q_3 = \frac{3(N+1)}{4} = \frac{3(64+1)}{4} = \frac{3 \times 65}{4} = 3 \times 16.25 = 48.75$ nearest *c.f.* to 52

Salary Rs. 24 corresponds to 52.

$\therefore Q_3 =$ Rs. 24

$$D_4 = \frac{4(N+1)}{10} = \frac{4 \times (64+1)}{10} = \frac{4 \times 65}{10} = \frac{260}{10} = 26\text{th}$$ observation nearest to *c.f.* 38

which corresponds to 22 $\therefore D_4 = 22$

$D_6 = \frac{6(N+1)}{10} = \frac{6\times(64+1)}{10} = \frac{6\times 65}{10} = \frac{390}{10} = 39\text{th}$ observation to c.f. 52 (higher)

which corresponds to 24; $\therefore D_6 = 24$

$P_{30} = \frac{30(N+1)}{100} = \frac{30\times(64+1)}{100} = \frac{3\times 65}{10} = \frac{195}{10} = 19.5\text{th}$ observation nearest to *c.f.* 22

which corresponds to 18, $\therefore P_{30} = 23$

$P_{70} = \frac{70(N+1)}{100} = \frac{70\times(64+1)}{100} = \frac{7\times 65}{10} = \frac{455}{10} = 45.5\text{th}$ observation nearest to

c.f. 52 which corresponds to 24; $\therefore P_{70} = 24$

Example 38 : Calculate Q_1, Q_3, D_2, D_7, P_{45}, P_{80} from the following :

Class :	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
f:	7	15	18	25	30	20	16	7	2

Solution:

Table No. III.24

Class :	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
f:	7	15	18	25	30	20	16	7	2
c.f. :	7	22	40	65	95	115	131	138	140

Here, $N = \sum f = 140$

$Q_1 = \frac{N}{4} = \frac{140}{4} = 35$ nearest 40

$\therefore Q_1$ class is 30-40

$\therefore l = 30,\ h = 10,\ c.f. = 22,\ f = 18$

$$\therefore Q_1 = l + \frac{\frac{N}{4} - c.f}{f} \times h = 30 + \frac{35-22}{18} \times 10 = 30 + \frac{130}{18} = 30 + 7.2 = 37.2$$

For Q_3, $\frac{3N}{4} = \frac{3\times 140}{4} = 3\times 35 = 105$ nearest to 115

$\therefore Q_3$ class is 60-70

$\therefore l = 60,\ h = 10,\ c.f. = 95,\ f = 20$

$$\therefore Q_3 = l + \frac{\frac{3N}{4} - c.f}{f} \times h = 60 + \frac{105-95}{20} \times 10 = 60 + \frac{10}{2} = 60 + 5 = 65$$

For $D_2, \frac{2N}{10} = \frac{2 \times 140}{10} = 28$ nearest to 40

$\therefore D_2$ class is 30-40

$\therefore\ l = 30,\ h = 10,\ c.f. = 22,\ f = 18$

$$\therefore\ D_2 = l + \frac{\frac{2N}{10} - c.f}{f} \times h = 30 + \frac{28-22}{18} \times 10 = 30 + \frac{60}{18} = 30 + 3.3 = 33.3$$

For $D_7, \frac{7N}{10} = \frac{7 \times 140}{10} = 98$ nearest to 115

$\therefore D_7$ class is 60-70

$\therefore l = 60, h = 10,\ c.f. = 95,\ f = 20$

$$\therefore D_7 = l + \frac{\frac{7N}{4} - c.f}{f} \times h = 60 + \frac{98-95}{20} \times 10 = 60 + \frac{30}{20} = 60 + 1.5 = 61.5$$

For $P_{45}, \frac{45 \times N}{100} = \frac{45 \times 140}{100} = 45 \times \frac{14}{10} = 63$ nearest to 65

$\therefore P_{45}$ class is 40-45

$\therefore l = 40, h = 10, c.f. = 40,\ f = 25$

$$\therefore P_{45} = l + \frac{\frac{45N}{100} - c.f}{f} \times h$$

$$= 40 + \frac{\frac{45 \times 140}{100} - 40}{25} \times 10$$

$$= 40 + \frac{63-40}{25} \times 10 = 40 + \frac{230}{25} = 40 + 9.2 = 49.2$$

For $P_{80}, \frac{80N}{100} = \frac{80 \times 140}{100} = 112$ nearest to 115

$\therefore P_{80}$ class is 60-70

$\therefore l = 60, h = 10, c.f. = 95, f = 20$

$$\therefore P_{80} = l + \frac{\frac{80N}{100} - c.f}{f} \times h = 60 + \frac{112 - 95}{20} \times 10 = 60 + \frac{7}{2} = 60 + 3.5 = 63.5$$

Exercise : III

1. What do you mean by statistical average? What are the different types of measure of central tendency or average?
2. What are the characteristics of an ideal average? Mention the uses of A.M., G.M., and H.M.
3. Define Arithmetic mean, Geometric mean and Harmonic mean. State their merits and demerits.
4. Define Arithmetic mean, median and mode. Discuss their relative advantages and limitations.
5. Show that $A.M. \geq G.M. \geq H.M.$
6. What are the different types of mean? Discuss the advantages and limitations of arithmetic mean.
7. Define median. Write the advantages and limitation of median.
8. What are the different types of mean? Write a note on the importance of arithmetic mean.
9. Define arithmetic mean. Write a note on importance of arithmetic mean.
10. Write two important objectives of measures of central value.
11. Define weighted arithmetic mean. Write two uses of it.
12. Write two uses each of median and mode.
13. Write the different properties of Arithmetic mean.
14. Define quartiles, deciles and percentile.
15. If $\bar{x}_1$ and $\bar{x}_2$ be the A.M. of the two series consisting n_1 and n_2 terms respectively then show that the mean of combined series is $(n_1 + n_2)$ items is $\frac{n_1\bar{x}_2 + n_2\bar{x}_2}{n_1 + n_2}$.
16. Choose the correct answer from the following :

(i) Which average is affected by extreme values -

(a) A.M. (b) G.M. (c) H.M. (d) Median

(ii) Which of the following relations does not hold -

(a) D_5 = Median (b) P_{60} = Median

(c) P_{50} = Median (d) Q_2 = Median

(iii) Which is more than Q_2 and D_6 -

(a) D_8 (b) P_{75} (c) D_7 (d) all the above

(iv) Which is called average of position -

(a) Mode (b) Median (c) G.M. (d) H.M.

(v) Which of the following is less than Q_2 -

(a) D_4 and P_{35} (b) D_8 and P_{70} (c) Q_3 and D_6 (d) None

(vi) For qualitative data, the best average is -

(a) Mode (b) Median (c) G.M. (d) A.M/

(vii) Which of the following deciles are less than first quartile :

(a) D_2 and D_2 (b) D_2 and D_3 (c) D_1, D_2 and D_3 (d) None of the above

(viii) In which of the following average arrangement of data is necessary?

(a) A.M. (b) H.M. (c) Median (d) Mode

(ix) The correct relation among mean, median and mode is

(a) Mode – Mean = 3 (mean – median)

(b) Mean – Mode = 3 (Mean – Median)

(c) Mean – Median = 3 (Mean – Mode)

(d) None of these

(x) The sum of deviations of the observations is zero from :

(a) Mode (b) Median (c) A.M. (d) None

[Ans: (i)-a, (ii)-(b), (iii)-(d), (iv)-(b), (v)-(a), (vi)-(b), (vii)-(a), (viii)-(c), (ix)-(b), (x)-(c)]

17. Fill in the blanks:

(a) .... is not affected by extreme observations.

(b) Geometric Mean is the .... of A.M. and H.M.

(c) In the calculation of ...., all the observations are taken into consideration.

(d) .... is very much affected by extreme values.

(e) The ogives is 'less than type' and 'more than type' intersect at ....

(f) The mode of a distribution is the value that has the highest ....

(g) The sum of deviations of the observations from their A.M. is ....

(h) If any one of the observations is zero .... will be equal to zero.

(i) .... may be calculated from a frequency distribution with open end classes.

(j) If any one of the observation is zero .... cannot be calculated.

(k) If the mean of the series $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ is $\bar{x}$, the mean of the series $\frac{x_1}{5}, \frac{x_2}{5}, \frac{x_3}{5} \ldots\ldots\ldots\ldots \frac{x_n}{5}$ ....

(l) Average suited for qualitative phenomena is ....

[Ans: (a) Median, (b) G.M., (c) Mean, (d) A.M., (e) Median, (f) frequency, (g) zero, (h) G.M., (i) Median, (j) H.M., (k) $\frac{\bar{x}}{5}$, (l) median]

18. Select the following statement whether True (T) or False (F) :

(i) A.M. is much affected by extreme values.

(ii) Median is not affected by extreme values.

(iii) H.M. cannot be calculated if any value is zero.

(iv) Median is computed measure of average.

(v) If any one of the observations is zero, G.M. will be zero.

(vi) Q_1 is less than D_2 and D_3.

(vii) Mode have more than one value.

(viii) Geometric mean is the average of A.M. and H.M.

(ix) P_{50} is not equal to median.

(x) Mean may be located graphically from histogram.

[Ans: (i) T, (ii) T, (iii) T, (iv) F, (v) T, (vi) F, (vii) T, (viii) F, (ix) F, (x) F]

19. Daily wages of 6 workers are Rs. 70, 42, 85, 75, 68, 55. Find A.M. and G.M.

[Ans: Rs. 65.83, Rs. 64.21]

20. Find A.M., G.M. and H.M. of the numbers 3, 6, 24 and 48.

[Ans: 20.25, 12, 7.11]

21. Find x when A.M. of 7, $x - 2$ and $x + 3$ is 9.

[Ans: 9.5]

22. Marks obtained by 30 candidates in a certain test are given below. Find average marks.

No. of students :	4	2	3	5	7	5	4
Marks obtained :	50	55	63	70	71	80	91

[Ans: 70.33]

23. Find Mean, median and Mode :

Wts (kg) :	15	16	17	18	19	20
No. of items :	6	12	23	30	9	1

[Ans: Mean 17.33 kg, Median 17 kg, Mode 18 kg]

24. Find the average weekly wages from the following frequency distribution :

Wages (Rs.) :	30-40	40-50	50-60	60-70	70-80	80-90
No. of workers :	8	20	40	18	10	4

[Ans: Rs. 56.40]

25. A.M. of the following incomplete frequency distribution is 67.75 inch. Find the value of f_y.

Height (inch) :	60-62	63-65	66-68	69-71	72-74
Frequency	15	54	f_3	81	24

[Ans: 126]

26. The A.M. of the following distribution is 50. Find the missing frequencies:

Class :	0-20	20-40	40-60	60-80	80-100	Total
Frequency :	17	f_1	32	f_2	19	120

[Ans: $f_1 = 28, f_2 = 24$ **]**

27. Find A.M. from the following frequency distribution :

Marks above :	0	10	20	30	40	50
No. of students :	40	37	30	20	7	3

[Ans: 29.25 mark]

28. Construct a grouped frequency distribution and find mean :

Wages below (Rs.) :	10	20	30	40	50	60
No. of persons :	17	36	68	79	90	100

[Ans: A.M. = 26]

29. Find median from the following frequency distribution :

Marks :	0-10	10-20	20-40	40-60	60-90	90-100
No. of students :	5	15	20	12	8	4

[Ans: Median = 4, Mode = 4]

[Ans: 32 marks]

[Note: Widths of class intervals are unequal]

30. Find Mode :

Marks :	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45
No. of students :	7	10	16	32	24	18	10	5	45

[Ans: 18.83 marks]

31. Find Mean and Median and Mode by using empirical relation :

Wages :	130-134	135-139	140-144	145-149	150-154	155-159	160-164
No. of workers :	5	15	28	24	17	10	1

[Ans: Mean = 145, Median = 144.92, Mode = 144.08]

32. In the following frequency distribution one frequency is missing. The median of the distribution is 53.5. Find the missing frequency.

Class :	20-30	30-40	40-50	50-60	60-70	70-80
Frequency :	3	5	f_3	20	4	

[Ans: 12]

33. Find the Modal income of the following distribution :

Monthly income (Rs.) :	1000-1500	1500-2000	2000-2500	2500-3000	3000-3500	3500-4000
Frequency :	30	50	75	68	43	24

[Ans: Rs. 2390.62]

34. The Median and Mode of the sum distributions are known as 27 and 26 respectively. Find the values of *a* and *b*.

[Ans: *a* = 8, *b* = 7]

35. Find Q_1, Q_3 and Mode from the following table :

Wages :	20-25	25-30	30-35	35-40	40-45	45-50	50-55	55-60
Persons :	5	70	100	180	150	120	70	60

[Ans: Q_1 = 35.35, Q_3 = 47.40, Mode = 38.64]

36. Find Q_1, Q_3, D_4 and P_{60} for the following distribution :

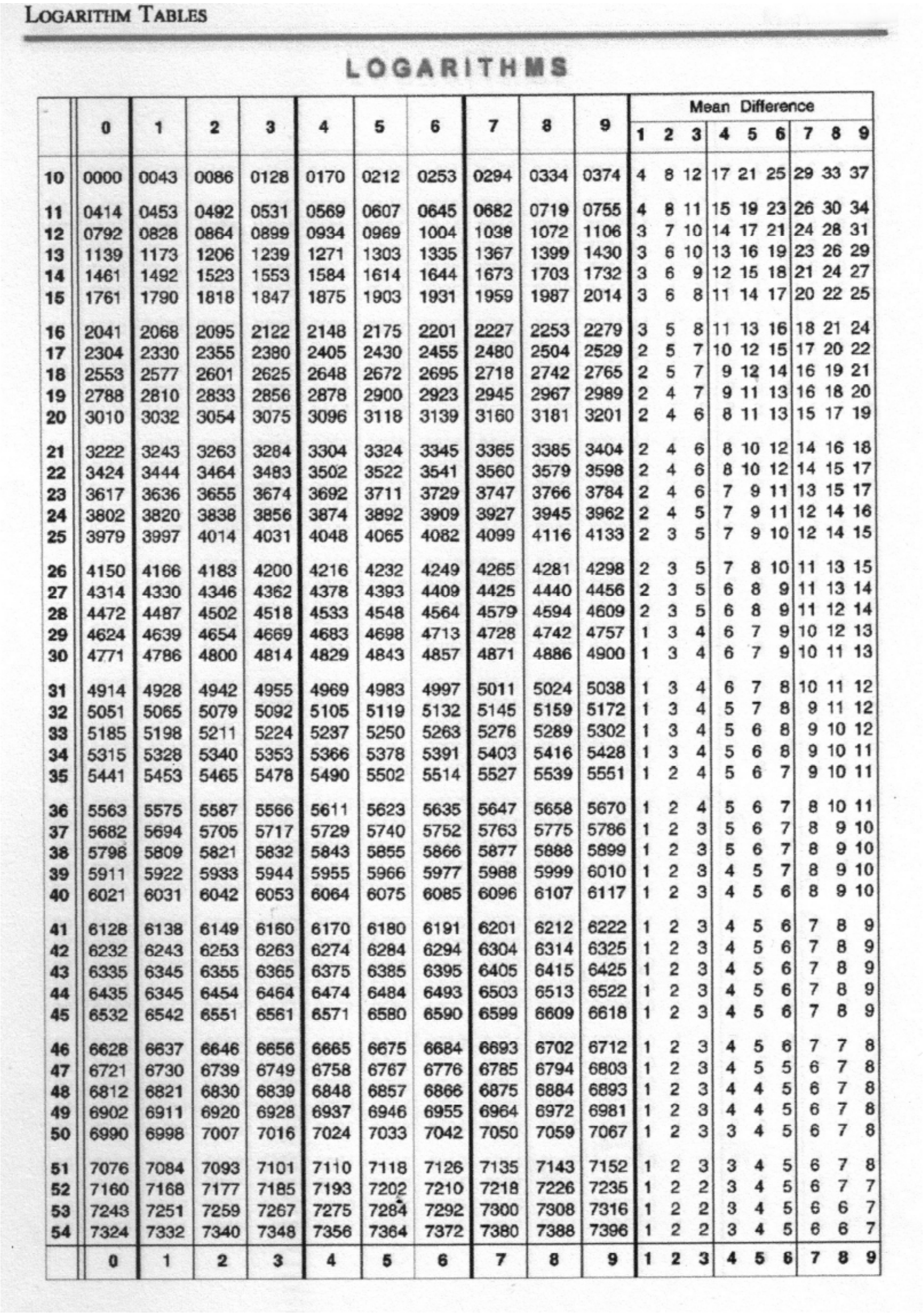
Marks :	10-15	15-20	20-25	25-30	30-35	35-40	40-45	45-50	50-55	55-60	60-65
Frequency :	5	6	12	47	39	38	28	25	17	14	13
Marks :	65-70	70-75									
Marks :	7	4									

[Ans: Q_1 = 29.34, Q_3 = 48.25, D_4 = 34.1 marks, P_{60} = 41.07 marks]

37. Find Q_1, Q_3, D_4, D_9, P_{43} and P_{78} from the data given below :

Weight (kg) :	30-39	40-49	50-59	60-69	70-79	80-89	90-99
No. of persons :	1	3	11	21	43	32	9

[Ans: Q_1 = 66.64 marks, Q_3 = 82.94 marks, D_4 = 72.2 marks, D_9 = 88.56 marks, P_{43} = 73.13 marks, P_{78} = 84.06 marks]



LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5566	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6345	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
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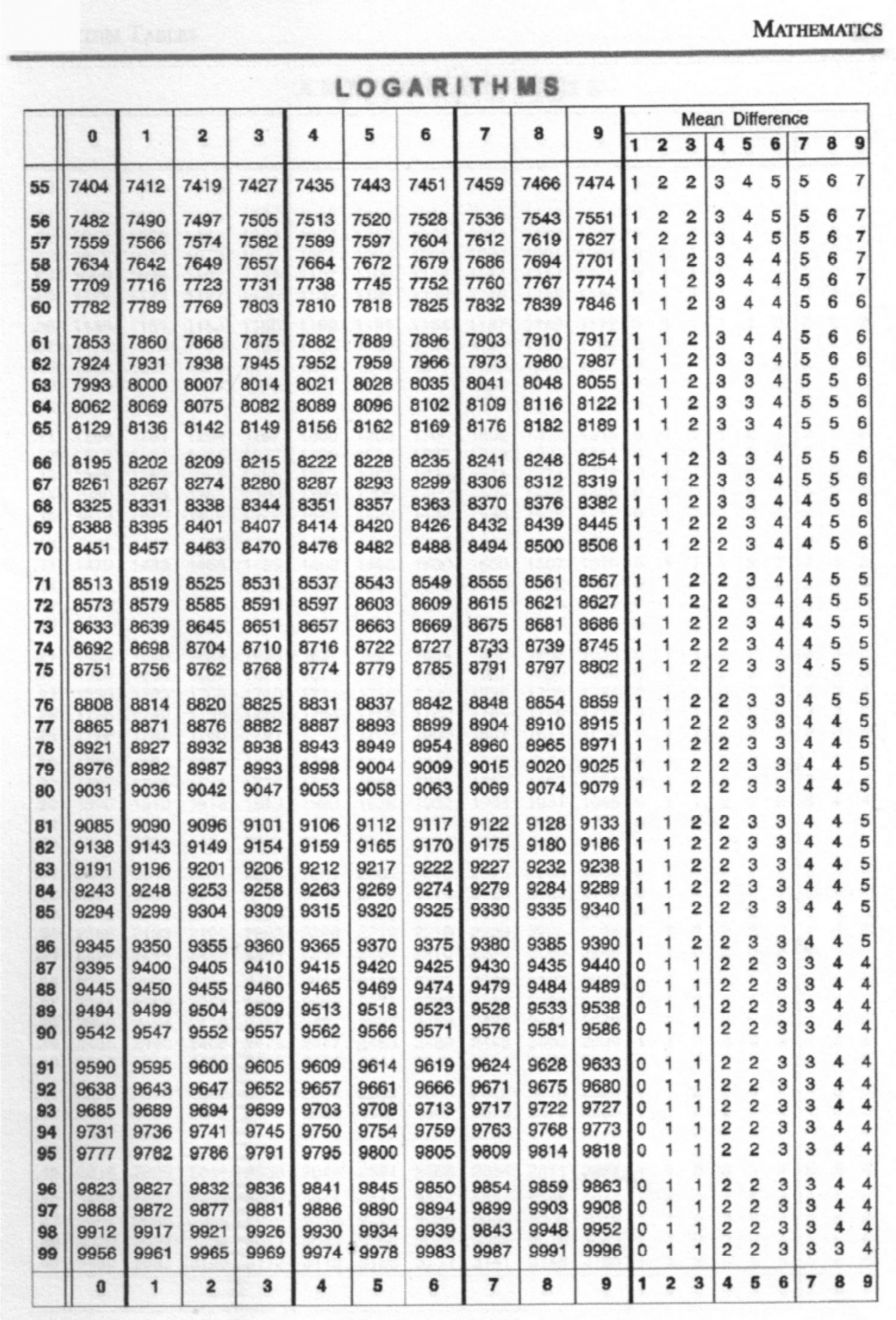

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
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56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7769	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9843	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1289	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
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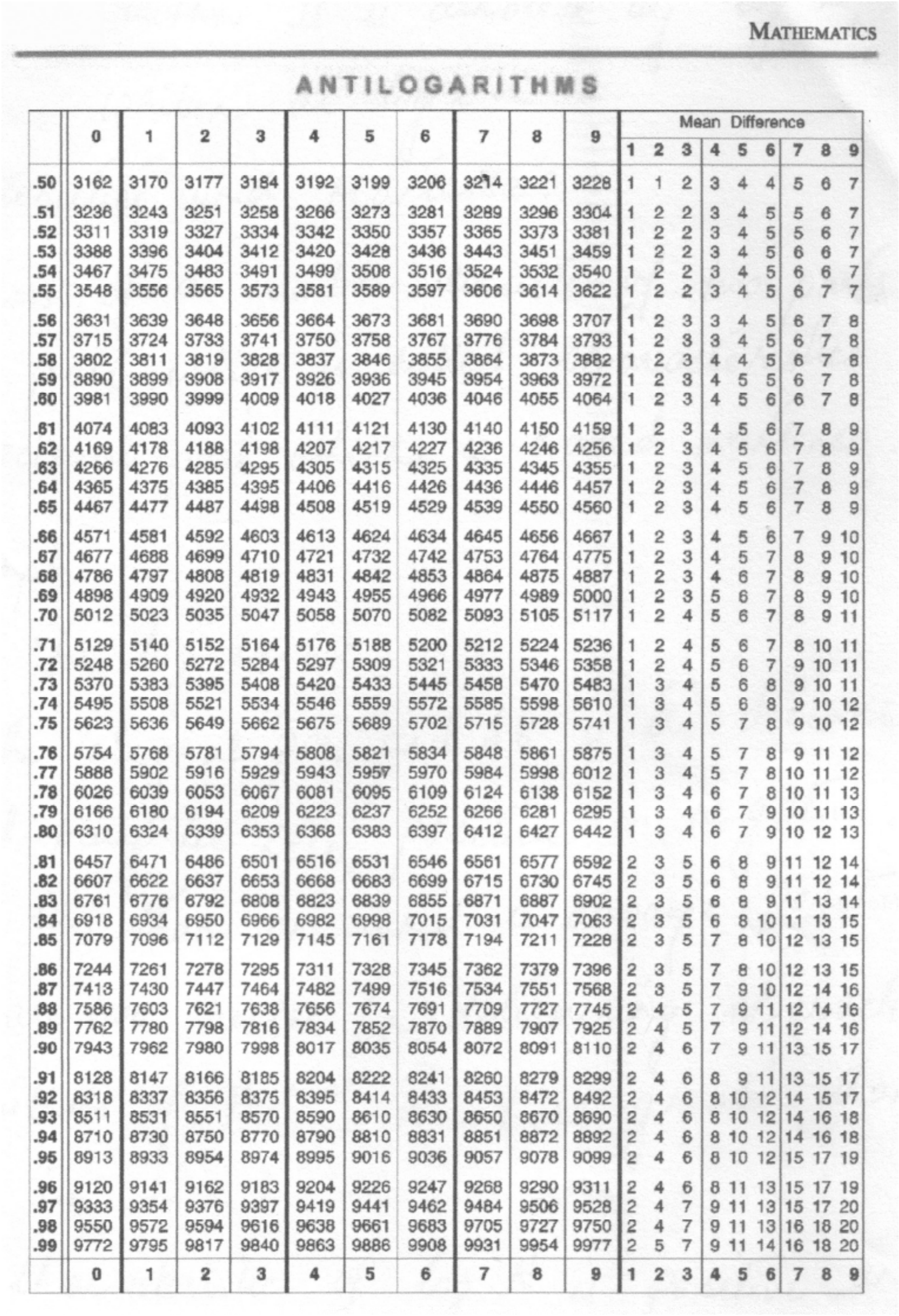

ANTILOGARITHMS

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.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6715	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
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.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
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.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9